Introducing Poset-based Connected n-Manifolds and $\mathcal{P}$-well-composedness in Partially Ordered Sets

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Abstract

In discrete topology, discrete surfaces are well-known for their strong topological and regularity properties. Their definition is recursive, and checking if a poset is a discrete surface is tractable. Their applications are numerous: when domain unicoherence is ensured, they lead access to the tree of shapes, and then to filtering in the shape space (shapings); they also lead to Laplacian zero-crossing extraction, to brain tumor segmentation, and many other applications related to mathematical morphology. They have many advantages in digital geometry and digital topology since discrete surfaces do not have any pinches (and then the underlying polyhedron of their geometric realization can be parameterized). However, contrary to topological manifolds known in continuous topology, discrete surfaces do not have any boundary, which is not always realizable in practice (finite hyper-rectangles cannot be discrete surfaces due to their non-empty boundary). For this reason, we propose the three following contributions: (1) we introduce a new definition of boundary, called border, based on the definition of discrete surfaces, and which allows us to delimit any partially ordered set whenever it is not embedded in a greater ambient space, (2) we introduce $\boldsymbol{\mathcal{P}}$-well-composedness similar to well-composedness in the sense of Alexandrov but based on borders, (3) we propose new (possibly geometrical) structures called (smooth) $\boldsymbol{n}$-PCM's which represent almost the same regularity as discrete surfaces and that are tractable thanks to their recursive definition, and (4) we prove several fundamental theorems relative to PCM's and their relations with discrete surfaces. We deeply believe that these new $\boldsymbol{n}$-dimensional structures are promising for the discrete topology and digital geometry fields.

Keywords: Discrete topology, discrete surfaces, posets, combinatorial manifolds, well-composedness

1 Introduction

In the world of Algebraic Topology [2–5], several discrete structures related to manifolds exist: n-pseudo-manifolds [6] are homogeneous connected n-dimensional simplicial complexes (see Figure 1), combinatorial (or equivalently stellar [7]) manifolds are pure simplicial complexes (see Figure 2) where the link of every vertex is either a combinatorial/stellar $(n-1)$-ball (boundary case) or a combinatorial/stellar $(n-1)$-sphere (interior case). In the context of partially ordered sets (or *posets*), a counterpart of this manifolds exists and is said to be a (discrete) n-surface [8]. In fact, assuming that we work only with simplicial complexes, there exists a classification theorem [6] which tells us that every combinatorial n-manifold is an n-surface, and every n-surface is an n-pseudo-manifold.

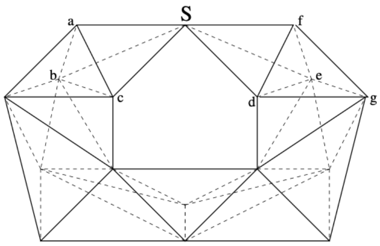


Fig. 1 The pinched sphere, well-known to be a pseudo-manifold but not a combinatorial manifold. Indeed, the link of the vertex S (at the pinch position), made of the closures of the triangles $\{a, b, c\}$ and $\{d, e, f\}$, is not connected. Consequently, this link is not a 1-sphere and the studied structure is not a combinatorial manifold. Note that the geometrical representation of this structure is 2D and in a 3D ambient space. This figure is extracted from [1].

On these discrete structures, there exist different ways to compute the boundaries. Assuming that we work with a complex [9, 10] of dimension n, the boundary is the closure of the $(n-1)$-faces which are faces of only one n-face of the complex. In Discrete Topology [11], it is common to work with n-D Khalimsky grids (denoted by $\mathbb{H}^n$) where Najman and Géraud [12] define the (combinatorial) boundary of some set X in $\mathbb{H}^n$ as the intersection of the closure of this set and the closure of its complementary in $\mathbb{H}^n$. For a combinatorial manifold, as detailed above, the boundary is the set of simplices of this manifold whose link is a combinatorial $(n-1)$-ball. Now, if we consider also continuous topology [13–15], the boundary of a topological manifold [16] is this same space minus its topological interior.

Many works are related to contours in computational geometry: see the works of Herman and Udupa [17] where they introduced the cuberille and the directed-contour representations, useful for quick manipulation and display of objects in 3D volumes; these works led to fast *surface tracking* algorithms. We will recall the works of Arcelli *et al.* [18] relative to contour tracing, the ones of Martinez *et al.* [19] and of Kwok *et al.* [20] relative to contour-based thinning, and the ones of Kerautret and Lachaud [21] where they proceed to robust curvature estimation. We can also refer to [6] coming from digital topology where Daragon *et al.* develop frontier orders (a Marching Cubes-like algorithm), based on discrete surfaces and to barycentric subdivisions to nicely separate sets. In [22], Alayrangues *et al.* show the equivalence between (a subclass of) n-G-maps used in geometric modeling and computational geometry and the discrete n-surfaces used in discrete topology. Note that the term *border* is not new in discrete topology, it has been introduced in [23] for n-G-maps and its goal is to optimize homology computation, reason for which it satisfies that $\partial\partial = 0$. These structures are routinely used in geometric modeling and computational geometry [24].

Note that discrete surfaces are well-known in Mathematical Morphology [25, 26] because they are strongly related to the tree of shapes [12, 27, 28]: a sufficient condition to obtain a well-defined tree of shapes is to have a well-composed [29–31] image (that is, whose boundaries are discrete surfaces) defined on a unicoherent domain as input. This leads then to many applications [32] as image filtering in the shape-space [33], connected filtering [34], object segmentation using context-based energy estimators [35], hierarchical segmentation [36], local feature detection [37], biomedical image segmentation [38], Laplacian zero-crossing extraction [39], blood vessels segmentation [40], and so on.

The provided paper is strongly related to a sequence of previously published articles [10, 30, 31, 41–48] since discrete surfaces are directly related to well-composedness. Their goal was to show the relations between the different flavours of well-composedness or to provide algorithms able to make a given set well-composed, always to benefit of nice topological properties (no pinches, no topological issues, separation properties, equivalence of connectivities, and so on). However, in all these works, we needed an ambient space to be able to define well-composedness like $\mathbb{Z}^n$ or the Khalimsky grid (both being infinite spaces), or even a discrete surface (which loop and has no boundary). In other words, well-composedness brings very nice topological properties but needs strong constraints on the ambient space. To get rid of these limitations, we propose a definition of *border* which does not need any ambient space and the one of well-composedness which follows from it ($\mathcal{P}$*-well-composedness*); these two concepts need only a topology that can be easily obtained just be choosing the inclusion relationship on the studied poset, making it an Alexandrov topological space. Thanks to these new concepts, we are then able to provide regular structures that

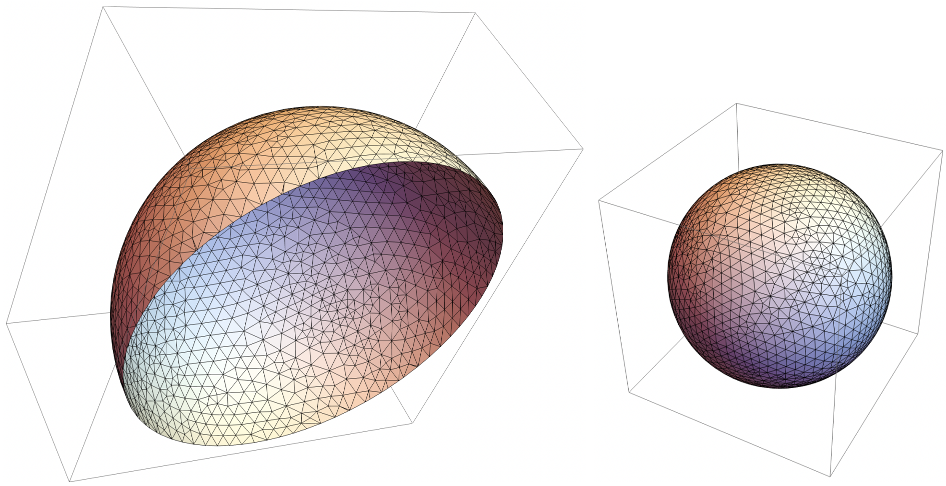


Fig. 2 Two combinatorial manifolds of dimension 2. On the left side, for any vertex of this structure, the link of a vertex in the structure is either a 1-sphere (interior), or a 1-ball (boundary). On the right side, for any vertex of this structure, the link of a vertex in this same structure is a 1-sphere (so this structure has no boundary).

we call *poset-based connected manifolds* (shortly n-PCM's), which show the same regularity properties as discrete surfaces but which have a border. Several fundamental theorems follow from these PCM's and show that they might become in the future a powerful tool in discrete topology and digital geometry.

So, the plan is the following. Section 2 recalls the necessary material in axiomatic digital topology. Section 3 introduces our definitions of borders in a poset and Section 4 introduces our definitions of $\mathcal{P}$-well-composedness, of n-PCMs, and of smooth n-PCMs. Section 5 presents our main theorems. Section 6 shows that joining PCMs and surfaces lead to PCMs, Section 8 concludes the paper, and Section A contains the formal proofs of our main theorems.

2 Axiomatic digital topology

For a, b two integers, we recall that $[\![a, b]\!]$ is defined as the intersection of $[a, b]$ with $\mathbb{Z}$, that is, it is the set of integers greater than or equal to a and lower than or equal to b.

2.1 Basics

For A and B two sets of arbitrary elements, $A \times B$ denotes the *Cartesian product* of A and B and is defined as $\{(a, b)\ ;\ \ a \in A,\ b \in B\}$.

Definition 1 (Binary and order relations [49]) A *binary relation* R defined on a set of arbitrary elements X is a subset of $X \times X$, and we denote by $x \in R(y)$ or equivalently $x\ R\ y$ the fact that $(x, y) \in R$. An *order relation* is a binary relation R which is reflexive, antisymmetric, and transitive.

We denote by $R^{\square}$ the binary relation on X defined such that, $\forall x, y \in X$, $\left\{x\ R^{\square}\ y\right\} \Leftrightarrow \{x\ R\ y \text{ and } x \neq y\}$. A set X of arbitrary elements supplied with an order relation R on X is called a *poset* and is denoted by (X, R), or shortly $|X|$ when no ambiguity is possible.

Definition 2 (Topology) Let X be a set of arbitrary elements, and let $\mathcal{U}$ be a set of subsets of X. We say that $\mathcal{U}$ is a *topology* on X if $\emptyset$ and X are elements of $\mathcal{U}$, if any union of elements of $\mathcal{U}$ are elements of $\mathcal{U}$, and if any finite intersection of elements of $\mathcal{U}$ is an element of $\mathcal{U}$. X supplied with $\mathcal{U}$ is denoted $(X, \mathcal{U})$ or shortly X and is called a *topological space*.

Definition 3 (Open and closed sets) The elements of $\mathcal{U}$ are then called the *open sets* of X and any complement of an open set in X is called a *closed set* of X. We say that a subset of X which contains an open set containing a point x is a *neighborhood* of x in X.

A topological space X is said *(topologically) connected* if it is not the disjoint union of two non-empty open sets.

Definition 4 (T_0-spaces) A *T_0-space* [50–52], let say X, is a topological space which satisfies the T_0 *axiom of separation*: for two distinct elements x, y of X, there exists a neighborhood of x in X which does not contain y or a neighborhood of y in X which does not contain x.

Definition 5 (Discrete T_0-spaces [53, 54]) A *discrete T_0-space* is a T_0-space space where any intersection of open sets is an open set.

Let $R : X \times Y \to \{0, 1\}$. We define the *inverse relation* R^{-1} of R as the relation $R^{-1} : Y \times X \to \{0, 1\}$ satisfying $bR^{-1}a$ iff aRb is true.

Remark 1 (From posets to topological spaces (Th. 6.52, p. 28 [50])) Posets can be considered as topological spaces in the sense that we can induce a topology on any poset based on its order relation: for a poset (X, R), the corresponding closed sets are the sets $C \subseteq X$ such that $\forall x \in C$, $R(x)$ is included in C. Open sets are the sets U such that for any $h \in U$, $R^{-1}(h)$ is included into U. Posets can be seen as discrete T_0-spaces and are generally called *Alexandrov (topological) spaces* [54].

Definition 6 (Combinatorial closures, openings, and neighborhoods) On an Alexandrov space $|X| = (X, R)$, for any element $h \in X$, we define respectively the *combinatorial closure* of h in X: $\alpha_X(h) := \{h' \in X\ ;\ \ h' \in R(h)\}$, its inverse operator called the *combinatorial opening* of h in X: $\beta_X(h) := \{h' \in X\ ;\ \ h \in R(h')\}$, and the *neighborhood* of h in X:

$$\theta_X(h) := \{h' \in X\ ;\ \ h' \in R(h) \text{ or } h \in R(h')\},$$

which leads to $\theta_X(h) = \alpha_X(h) \cup \beta_X(h)$.

Remark 2 (“Adjacency” in T_0 spaces.) Let a, b be two elements of X supplied with the order relation R. When the property aRb holds with $R =\subseteq$, we mean that “a is included in b”, which is a form of adjacency (imagine a 0-face $\{0\}$ contained in the 1-face $\{0, 1\}$). However, it is not symmetrical (the 1-face $\{0, 1\}$ is not contained in the 0-face $\{0\}$). In other words,“a is included in b” does not mean that we have “b included in a”. Indeed, both properties are true only when a and b are equal. All this is related to the T_0 separation axiom which is true for Alexandrov spaces like posets: for every two different elements a, b chosen in the poset X, their neighborhood are different, we say that they are *distinguishable*. At the contrary, *adjacency* is generally symmetrical. For two elements $a, b \in \mathbb{Z}^n$, a and b are said to be *$2n$-adjacent*, shortly c_{2n}, when they differ from one coordinate only, and that this difference is equal to one. These same elements are said to be *$(3^n - 1)$-adjacent*, shortly c_{3^n-1}, when they differ from at least one coordinate, and that the maximum of the absolute differences of these two coordinates is equal to one. In this discrete setting, saying that a and b are adjacent is equivalent to say that b and a are adjacent. That is why, in Alexandrov space, we use the θ operator which satisfies $a \in \theta(b)$ iff $b \in \theta(a)$, since $\theta = \alpha \bigcup \alpha^{-1} = \alpha \cup \beta$. When considering the operator $\theta^\square$, then $a \in \theta^\square_X(b)$ is equivalent to $b \in \theta^\square_X(a)$, and this two properties lead to $a \neq b$. Then, the property “$a \in \theta^\square_X(b)$” can be read “$a$ is adjacent to b in X”. However, for sake of simplicity, we will use *Daragon notations* [1] (α, β and θ).

Remark 3 Thanks to the properties explained before, for any $h \in X$, $\alpha_X(h)$ will be a closed set in X and each $\beta_X(h)$ will be an open set in X. In other words, combinatorial and topological definitions are equivalent in Alexandrov spaces.

The operators α, β and θ are also defined for sets: $\forall S \subseteq X$, $\alpha_X(S) := \cup_{p \in S}\alpha_X(p)$, $\beta_X(S) := \cup_{p \in S}\beta_X(p)$, and $\theta_X(S) := \cup_{p \in S}\theta_X(p)$, where $\alpha_X(S)$ is closed in X and $\beta(S)$ is open in X thanks to the properties exposed before.

Remark 4 Let A, B two subsets of a same poset X. Due to the symmetry of the operator θ, we obtain that $\theta_X(A) \cap B \neq \emptyset$ is equivalent to $A \cap \theta_X(B) \neq \emptyset$. Indeed, if we assume that $\theta_X(A) \cap B$ is non empty, there exists $a \in A$ such that $\theta_X(a) \cap B \neq \emptyset$. There exists then $b \in B \cap \theta_X(a)$. Since $b \in \theta_X(a)$, $a \in \theta_X(b)$, and then $\theta_X(b) \cap A \neq \emptyset$, then $\theta_X(B) \cap A \neq \emptyset$. For the inverse implication, let us swap A and B in the initial formula. We will obtain the converse implication. So, $\theta_X(A) \cap B \neq \emptyset$ is equivalent to $A \cap \theta_X(B) \neq \emptyset$.

Note that the operators α and β are *idempotent*: $\alpha \circ \alpha = \alpha$ and $\beta \circ \beta = \beta$. However, the neighborhood operator is not idempotent.

Definition 7 (Suborder [49]) Assuming that S is a subset of the domain X of a poset $|X| = (X, \alpha_X)$, the *suborder* of $|X|$ relative to S is the poset $|S| = (S, \alpha_S)$ with, for any $h \in S$, $\alpha_S(h) = \alpha_X(h) \cap S$, $\beta_S(h) = \beta_X(h) \cap S$, and $\theta_S(h) = \theta_X(h) \cap S$.

Definition 8 (Pointed closures, openings, and neighborhoods) Let X be some poset and let h be some of its elements. Referring to the notation $R^{\square}$ detailed before, we define the *pointed closure* of h in X:

$$\alpha_X^{\square}(h) = \alpha_X(h) \setminus \{h\},$$

the *pointed opening* of h in X:

$$\beta_X^{\square}(h) = \beta_X(h) \setminus \{h\},$$

and the *pointed neighborhood* of h in X:

$$\theta_X^{\square}(h) = \theta_X(h) \setminus \{h\}.$$

Definition 9 (Paths [49]) We call *path* into a set $S \subseteq X$ a finite sequence $(p^0, \ldots, p^k)$ such that for all $i \in [\![1, k]\!]$, $p^i \in \theta_S^{\square}(p^{i-1})$. We say that a digital set $S \subseteq X$ is *path-connected* [49] if for any points p, q in S, there exists a path into S joining them. Path-connectedness and topological connectedness are equivalent [49, 54] in $|S|$ like in any Alexandrov space.

Definition 10 (Connected components) The greatest (path-)connected set in the digital set $S \subset X$ containing $p \in X$ is called the *connected component* [50] of S containing p and we denote it by $\mathcal{CC}(S, p)$; by convention, when p does not belong to S, we write $\mathcal{CC}(S, p) = \emptyset$. Any non-empty subset of a poset S which can be written $\mathcal{CC}(S, p)$ for some $p \in S$ is called a *connected component* of S. The set of connected components of a poset S is denoted by $\mathcal{CC}(S)$.

Definition 11 (Rank) The *rank* of an element h in the suborder $|S|$ of X is denoted by $\mathrm{rk}(h, |S|)$ and is defined in a recursive fashion: it is equal to 0 if $\alpha_S^{\square}(h) = \emptyset$ and to $\max_{x \in \alpha_S^{\square}(h)}(\mathrm{rk}(x, |S|)) + 1$ otherwise. The *rank* of $|S|$ is denoted by $\mathrm{rk}(|S|)$ and is equal to the maximal rank of its elements.

An element h of S such that $\mathrm{rk}(h, |S|) = k$ is called a *k-face* [49] of S.

Definition 12 (CF-orders) Let $|S|$ be a suborder of a poset X, then $|S|$ is said to be *countable* if S is countable. Also, $|S|$ is called *locally finite* if for any element $h \in S$, the set $\theta_S(h)$ is finite. When $|S|$ is countable and locally finite, it is said to be a *CF-order* [49] in X.

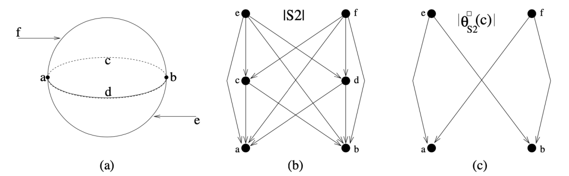


Fig. 3 A poset $|S_2| = (S_2, \subseteq)$ is depicted on the left side; it is made of the set $\{a, b, c, d, e, f\}$ of faces and supplied with a relation order $\subseteq$: f and e, represented by two half-spheres, contain both the two half-arcs d and c, which contain both the points a and b. On the Hasse diagram, showing the inclusion relationships between the elements of S_2 by using arrows, of $|S_2|$ is plotted, we see that the maximal rank of its elements is 2, thus this is a poset of rank 2. Now, this poset is connected, and for any element $h \in S_2$, the property "$\theta_{S_2}^{\square}(h)$ is a 1-surface" is satisfied, so $|S_2|$ is a 2-surface. This figure has been extracted from [1].

Definition 13 (Discrete surfaces) Let $|S|$ be a CF-order in a poset $|X|$; $|S|$ is said to be a (-1)-*surface* if $S = \emptyset$, or a 0-*surface* if S is made of two different faces $x, y \in X$ such that $x \notin \theta^{\square}(y)$, or a k-*surface*, $k \in [\![1, n]\!]$, if $|S|$ is connected and for any $h \in S$, $|\theta_S^{\square}(h)|$ is a $(k-1)$-surface.

Remark 5 (Discrete surfaces examples and counter-example) Some examples of discrete surfaces are

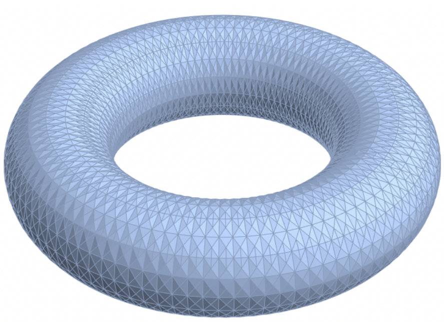


Fig. 4 The geometric realization of a discrete 2-surface drawing a 2-torus.

depicted here: see Figure 3 for a poset of rank 2, and see Figure 4 for a (geometric) 2-torus in a 3D space. In both cases, for any face h in the described poset X, the poset $|\theta_X^\square(h)|$ is a 1-surface (a simple closed curve). An example of poset which is not a 2-surface in the pinched sphere (see Figure 1), where we obtain at the pinch p that $|\theta_X^\square(p)|$ is made of two separated 1-surfaces, and thus it is not a 1-surface (since it is not connected).

Definition 14 (Khalimsky grids) Let $\mathbb{H}_0^1$ be the set $\{\{x\}\ ;\ x \in \mathbb{Z}\}$, let $\mathbb{H}_1^1$ be the set $\{\{x, x+1\}\ ;\ x \in \mathbb{Z}\}$. We define as $1D$ Khalimsky grid the set $\mathbb{H}^1 = \mathbb{H}_0^1 \cup \mathbb{H}_1^1$ and as n-D Khalimsky grid:

$$\mathbb{H}^n = \{h_1 \times \cdots \times h_n\ ;\ \forall i \in [1, n], h_i \in \mathbb{H}^1\}.$$

Intuitively, Khalimsky grids are *cubical grids* supplied with a topology. For example, in 3D, we consider all the unitary 3D cubes whose centers belong to $\mathbb{Z}^3$, and we take into account not only the cubes but also their 2D faces, their 1D edges, and their 0D corners. This way, we obtain a set of geometrical objects. Now, if we consider an operator α which provides for any geometrical object all the geometrical objects included into it, we have defined a topological operator called "closure". By grouping this set of geometrical objects and this closure operator, we have built the so called Khalimsky grid.

According to Evako *et al.* [55], the n-D Khalimsky grid $|\mathbb{H}^n|$ is an n-surface. Examples of 2-surfaces are given in Figures 3 and Figure 4.

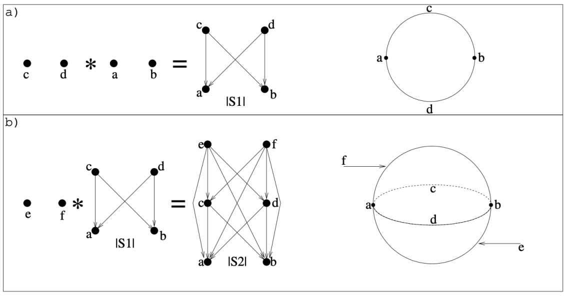


Fig. 5 Joins of posets: on the top, we can observe that joining two 0-surfaces lead to a 1-surface, that can be represented as a circle; on the bottom, we observe that the join of a 0-surface with a 1-surface leads to a 2-surface, that can be represented as a sphere. This figure is extracted from [1].

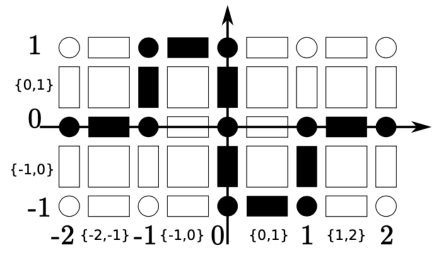


Fig. 6 A plain map from $\mathbb{H}^1$ to $\mathbb{H}^1$.This figure has been extracted from [12].

Definition 15 (Joins) Let $|X| := (X, R_X)$ and $|Y| := (Y, R_Y)$ be two posets; it is said that $|X|$ and $|Y|$ can be *joined* [49] if $X \cap Y = \emptyset$. If $|X|$ and $|Y|$ can be joined, the *join* of $|X|$ and $|Y|$ is denoted $|X|*|Y|$ and is equal to:

$$(X \cup Y, R_X \cup R_Y \cup X \times Y).$$

Some examples of join operations are described in Figure 5.

Proposition 1 ([1]) *Let $|X|$ and $|Y|$ be two posets that can be joined. The poset $|X|*|Y|$ is an $(n+1)$-surface with $n \in \mathbb{Z}$ iff there exists some integer $p \in [\![-1, n+1]\!]$ such that $|X|$ is a p-surface and $|Y|$ is an $(n-p)$-surface.*

Proposition 2 (Property 10 in [1]) *Let $|S|$ be a suborder of an n-surface $|X|$ with $n \geq 0$. Then $|S|$ is an n-surface iff for any $h \in S$, $|\alpha_S^\square(h)|$ is a $(k-1)$-surface and $|\beta_S^\square(h)|$ is an $(n-k-1)$-surface with $k = \text{rk}(h, |S|)$.*

Proposition 3 ([47, 48]) *When A, B are both n-surfaces, with $n \geq 0$, and satisfy $A \subseteq B$, then $A = B$.*

Definition 16 (Separated/disjoint unions) Let A, B be two subsets of a poset X. We call disjoint union of A and B the set $A \cup B$, assuming that $A \cap B = \emptyset$; we will denote it by $A \sqcup B$. We call *separated union* of A and B the set $A \cup B$, assuming that $A \cap \theta_X(B) = \emptyset$ (or equivalently $B \cap \theta_X(A) = \emptyset$). A separated union is then a disjoint union, but the converse is not necessarily true.

2.2 Plain maps

This subsection is not necessary for the proofs, it is only devoted to give the intuition of what is an image called a *plain map* (in a formal setting, many definitions are necessary to be able to define a plain map but it is not the goal of this paper).

Plain maps are *discrete functions with continuity properties* [12, 56] (see Figure 6). In particular, they satisfy the intermediate-value theorem. They can be defined as mappings from any poset to $\mathbb{H}^1$ but also to $\mathbb{Z}$ since we can easily define a bijection from $\mathbb{H}^1$ to $\mathbb{Z}$.

Plain maps are generally continuous representations of discrete images obtained thanks to *interpolations/immersions* [12, 31, 46, 57].

Finally, starting from any discrete mapping defined on discrete (topological or not) space, we will be able to compute their tree of shapes (when some particular properties are satisfied).

2.3 Threshold sets and the tree of shapes

Remark 6 As said before, we can define a bijection between $\mathbb{Z}$ and $\mathbb{H}^1$. Let us assume the following order in $\mathbb{H}^1$: $\dots < \{0\} < \{0,1\} < \{1\} < \{1,2\} < \dots$. Now we define $f : \mathbb{H}^1 \to \mathbb{Z} : h \to 2 * \text{mean}(h)$. This bijection preserves the order and then is an isomorphism between $\mathbb{H}^1$ and $\mathbb{Z}$. This means that they are *isomorphic* and we can identify them. In other words, instead of speaking about $\mathbb{H}^1$, we can speak about $\mathbb{Z}$ for the space value of the plain maps.

For any $\lambda \in \mathbb{Z}$, we call open *upper/lower threshold set* of a plain map $U : X \rightsquigarrow \mathbb{Z}$ the sets:

$$[U \rhd \lambda] = \{x \in X \; ; \; \forall v \in U(x), v > \lambda\},$$

$$[U \lhd \lambda] = \{x \in X \; ; \; \forall v \in U(x), v < \lambda\}.$$

In the same manner, for any $\lambda \in \mathbb{Z}$, we call closed *upper/lower threshold set* of a plain map $U : X \rightsquigarrow \mathbb{Z}$ the sets:

$$[U \unrhd \lambda] = \{x \in X \; ; \; \exists v \in U(x), v \geq \lambda\},$$

$$[U \unlhd \lambda] = \{x \in X \; ; \; \exists v \in U(x), v \leq \lambda\}.$$

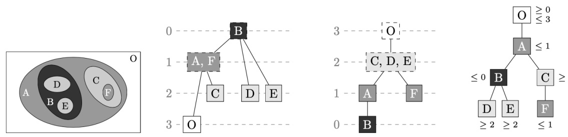


Fig. 7 The tree of shapes (on the right side) of the image drawn on the left side is the fusion of the min-tree (third column) whose leaves are minima and the max-tree (second column) whose leaves are maxima. The tree of shapes is a natural way to decompose a gray level image into components named *shapes*. This figure has been extracted from [58].

These thresholds sets satisfy the following property for any $\lambda \in \mathbb{Z}$:

$$[U \rhd \lambda] \sqcup [U \unlhd \lambda] = X = [U \lhd \lambda] \sqcup [U \unrhd \lambda].$$

Remark 7 Let X be some poset and let $U : X \rightsquigarrow \mathbb{Z}$ be a plain map. The closed threshold sets of U are closed under inclusion. Indeed, for any $h \in [U \unlhd \lambda]$, there exists some value $v \in \mathbb{Z}$ such that $U(h) \ni v \leq \lambda$. For any face $h' \in \alpha^{\square}_X(h)$, $U(h') \supseteq U(h)$ by property of a plain map, so $U(h')$ contains v satisfying $v \leq \lambda$. So, $h' \in [U \unlhd \lambda]$. The same reasoning applies for the dual closed threshold set.

For a given $\lambda \in \mathbb{Z}$, we can then define the *upper/lower shapes* of U of level λ:

$$\mathcal{S}_{\rhd\lambda} = \{\text{Sat}(\Gamma, p_\infty) \; ; \; \Gamma \in \mathcal{CC}([U \rhd \lambda])\},$$

$$\mathcal{S}_{\lhd\lambda} = \{\text{Sat}(\Gamma, p_\infty) \; ; \; \Gamma \in \mathcal{CC}([U \lhd \lambda])\},$$

with $\text{Sat}(., p_\infty)$ the *saturation operator* (also called the cavity fill-in operator) defined as: $\text{Sat}(\Gamma, p_\infty) = X \setminus \mathcal{CC}(X \setminus \Gamma, p_\infty)$ where $p_\infty \in X$ is the *exterior point*.

Definition 17 (Combinatorial boundary [12]) Let Y be some poset and let $X \subseteq Y$ be some suborder of Y. Then we define the *combinatorial boundary* of X the term:

$$\partial X = \alpha_Y(X) \cap \alpha_Y(Y \setminus X).$$

Definition 18 ([12]) Let $U : X \rightsquigarrow \mathbb{Z}$ be an plain map on X. We say that U is *Alexandrov well-composed* (AWC) when its combinatorial boundary is a separated union of discrete $(n-1)$-surfaces.

We recall that a tree is a set $\mathfrak{T}$ of arbitrary sets where for any $A, B \in \mathfrak{T}$, A and B are disjoint or nested.

Theorem 4 ([12]) *Let X be a unicoherent discrete topological space. Let U be an AWC plain map on X, then the set of shapes $\mathfrak{T}(U) = \{\mathcal{S}_{\lhd\lambda}\}_\lambda \cup \{\mathcal{S}_{\rhd\lambda}\}_\lambda$ is a tree.*

An example of tree of shape is depicted in the continuous case in Figure 7 to give the intuition of how it is built.

3 Borders, $\mathcal{P}$-well-composedness, n-PCMs and smooth n-PCMs

3.1 Border and interior

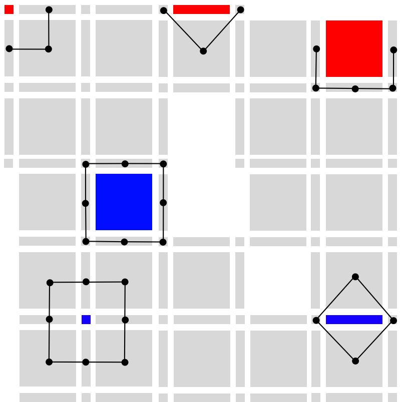


Fig. 8 Differentiation between border and interior points in any poset X of rank 2: the red faces h admit a neighborhood $|\theta_X^\square(h)|$ which is not a 1-surface, thus they belong to the border of the poset. At the contrary, the blue faces h' admit as neighborhood $|\theta_X^\square(h')|$ a simple closed curve, that is, a 1-surface. Consequently, they belong to the interior of the poset.

Let us now propose our definition of *border* in partially ordered sets.

Definition 19 (Border and Interior) Let X be a poset of rank $n \geq 0$. We define the border of X as the set:

$$\Delta X = \left\{ h \in X \; ; \; \left|\theta_X^\square(h)\right| \text{ is not a } (n-1)\text{-surface} \right\}.$$

Let X be a poset of rank $n \geq 0$. The *interior* of X is defined as the set:

$$\mathrm{Int}(X) := X \setminus \Delta X.$$

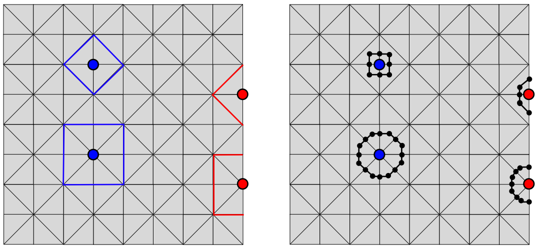


Fig. 9 Comparison between the boundary of a combinatorial manifold and the border of a poset. As depicted on the 2D combinatorial manifold on the left side, the 0-vertices admit a *link* which is either a simple closed curve (the blue points belong then to the interior of the manifold), or a simple (not closed) path (the red points belong then to the boundary of the manifold). The principle of border in a 2D poset is almost the same: on the right side, we have a poset where the vertices in blue admit as *neighborhood* a simple closed curve, that is, a 1-surface (they belong then to the interior of the poset); however, the vertices in red do not admit as *neighborhood* a simple closed curve but a simple path (they belong then to the border of the poset). As we will see later, the notion of border concerns not only the vertices but all the faces of the poset.

We show how to differentiate interior points from border points in Figure 8. We provide also a comparison of the computation of a border of a poset with the boundary of a combinatorial manifold in Figure 9.

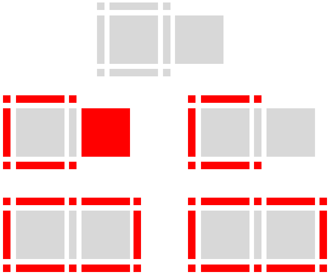


Fig. 10 Different types of boundaries (in red) on a same poset. In the raster scan order: a poset represented in the 2D Khalimsky grid, its border according to our definition, its homological boundary $\alpha(X')$ where X' is made of the 1-faces belonging to only one 2-face of the poset, its combinatorial boundary $\alpha(X) \cap \alpha(\mathbb{H}^2 \setminus X)$, its topological boundary $\alpha(X) \setminus \mathrm{Int}(X)$.

An example of border is depicted in Figure 10. The detailed computation of the border described in this paper is given in Figure 11.

Remark 8 When $|X|$ is a suborder of the poset $|Y|$ of rank $n \geq 0$, the border of (respectively the interior of)

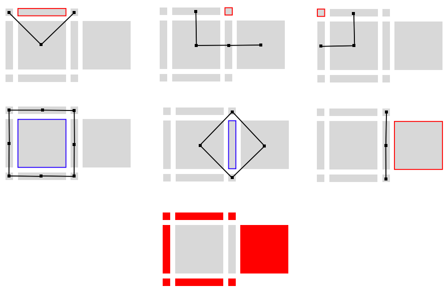


Fig. 11 The different steps used to compute the border of a given poset of rank 2: when the neighborhood (see the black little squares) of a face (encircled in red or blue) is not a 1-surface (*i.e.* a simple closed curve), then this face belongs to the border; otherwise, it does not.

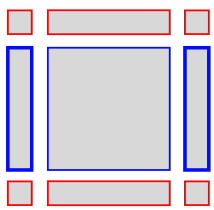


Fig. 12 The border of a poset $|X|$ suborder or a greater poset $|Y|$ is not always closed. The poset $|Y|$ is made of the nine faces presented above and is supplied with the inclusion relationship. The poset $|X|$ is depicted in blue, has a rank of 1, and thus the elements of X which belong to ΔX are the faces h satisfying that $|\theta_X^{\square}(h)|$ is not a 0-surface. ΔX is then made of the two 1-faces of Y. ΔX is not closed under inclusion in Y.

$|X|$ is not necessarily closed (respectively open) in $|Y|$ (see Figure 12).

Remark 9 When X is an n-surface, then $\Delta X = \emptyset$.

3.2 Coherence

Definition 20 We say that a poset $|X|$ is *coherent* when it is empty (case $n = -1$) or when for any $h \in X$, we have the following properties:

$$\begin{cases} \mathrm{rk}\left(\left|\theta_X^{\square}(h)\right|\right) = \mathrm{rk}(|X|) - 1, \\ \text{and } |\theta_X^{\square}(h)| \text{ is coherent.} \end{cases}$$

Remark 10 Discrete n-surfaces are coherent posets.

3.3 PWCness

Definition 21 ($\mathcal{P}$-well-composedness) Let $|X|$ be a poset. We say that this poset is *$\mathcal{P}$-well-composed* (PWC) when its border is a separated union of discrete $(n-1)$-surfaces.

Note that this definition is close to the *well-composedness in the sense of Alexandrov (AWC-ness)* [48] (AWC), the difference being that AWC-ness is related to combinatorial boundaries (which do need an ambient space) whereas $\mathcal{P}$-well-composedness is related to borders (which do not need an ambient space).

Proposition 5 *Let Y be a discrete n-surface, and X be a suborder of Y of rank n which is closed under inclusion. Then, $\Delta X = \partial X$. In other words, X is $\mathcal{P}$-well-composed iff it is AWC.*

Proof: Let h be some element of ΔX, that is, $|\theta_X^{\square}(h)|$ is not an $(n-1)$-surface. Since X is closed under inclusion, we have $|\alpha_X^{\square}(h)| = |\alpha_Y^{\square}(h)|$, which is a $(dim(h)-1)$-surface. Thus, the fact that $|\theta_X^{\square}(h)|$ is not an $(n-1)$-surface implies that $|\beta_X^{\square}(h)|$ is not an $(n-dim(h)-1)$-surface. At the contrary, $|\beta_Y^{\square}(h)|$ is an $(n-dim(h)-1)$-surface since Y is an n-surface. In other words, $\beta_X^{\square}(h) \subsetneq \beta_Y^{\square}(h)$ with $X \subseteq Y$. This implies that there exists some $y \in \beta_Y^{\square}(h) \setminus \beta_X^{\square}(h) = \beta_{Y\setminus X}^{\square}(h)$. Consequently, h belongs at the same time to $\alpha_Y(X)$ and to $\alpha_Y(Y \setminus X)$, and thus belongs to ∂X.

Conversely, when h is an element of ∂X, there exists $x \in X$ and $y \in Y \setminus X$ such that $h \in \alpha_Y(x) \cap \alpha_Y(y)$. Thus, x and y belong to $\beta_Y^{\square}(h)$, with $y \notin \beta_X^{\square}(h)$. Since $|\beta_Y^{\square}(h)|$ is an $(n-dim(h)-1)$-surface and $\beta_X^{\square}(h) \subsetneq \beta_Y^{\square}(h)$, $|\beta_X^{\square}(h)|$ is not an $(n-dim(h)-1)$-surface (by Proposition 3). The direct consequence is that $|\theta_X^{\square}(h)|$ is not an $(n-1)$-surface, that is, h belongs to ΔX. □

Definition 22 Let X be a poset. We say that a plain map $U : X \rightsquigarrow \mathbb{Z}$ is *$\mathcal{P}$-well-composed* when all its (closed) threshold sets are $\mathcal{P}$-well-composed.

Proposition 6 *Let X be a discrete n-surface. The plain map $U : X \rightsquigarrow \mathbb{Z}$ is* $\mathcal{P}$-well-composed *iff it is well-composed in the sense of Alexandrov.*

Proof: When U is PWC, all its closed threshold sets are PWC, that is, for any $\lambda \in \mathbb{Z}$, $\Delta[U \trianglerighteq \lambda]$ and $\Delta[U \trianglelefteq \lambda]$ are separated unions of discrete $(n-1)$-surfaces. Since $[U \trianglerighteq \lambda]$ and $[U \trianglelefteq \lambda]$ are closed under inclusion, it is equivalent by Proposition 5 to say that $\partial[U \trianglerighteq \lambda]$ and $\partial[U \trianglelefteq \lambda]$ are made of separated discrete $(n-1)$-surfaces. Since we have the following properties: $\partial[U \trianglerighteq \lambda] = \partial[U \triangleleft \lambda]$ and $\partial[U \trianglelefteq \lambda] = \partial[U \triangleright \lambda]$, AWCness and PWCness are then equivalent. This concludes the proof. □

As a corollary of Theorem 4, we obtain the following proposition, which shows that the results from [12] hold for PWC plain maps.

Proposition 7 *Let X be a unicoherent discrete topological space. Let U be a PWC plain map on X, then the set of shapes $\mathfrak{T}(U) = \{\mathcal{S}_{\triangleleft\lambda}\}_\lambda \cup \{\mathcal{S}_{\triangleright\lambda}\}_\lambda$ is a tree.*

Furthermore, it has been proven in [48] that AWCness and digital well-composedness (DWCness)[1] are equivalent on n-D cubical grids or complexes. This leads directly to the following proposition.

Proposition 8 *Let $n \geq 1$ be some integer. The plain map $U : \mathbb{H}^n \rightsquigarrow \mathbb{Z}$ is $\mathcal{P}$-well-composed iff it is digitally well-composed. In other words, PWCness and DWCness are equivalent on cubical grids.*

3.4 Remarkable property of the border

Proposition 9 *Let X be a coherent poset of rank $n \geq 0$, and h be an element of ΔX. Then we have the following remarkable property:*

$$\Delta\theta^{\square}_X(h) \subseteq \theta^{\square}_{\Delta X}(h).$$

In other words, the border of the neighborhood is included in the neighborhood of the border.

Proof: Let h be an element of ΔX, and h_0 be an element of $\Delta(\theta^{\square}_X(h))$ (and then h_0 is an element of X). We want to show that h_0 belongs to ΔX. By Definition 19, $\left|\theta^{\square}_{\theta^{\square}_X(h)}(h_0)\right|$ is not a $\left(\mathrm{rk}\left(\left|\theta^{\square}_X(h)\right|\right)-1\right)$-surface, that is, $\left|\theta^{\square}_{\theta^{\square}_X(h)}(h_0)\right|$ is not a $\left(\mathrm{rk}\left(X\right)-2\right)$-surface, by coherence of X. Now let us assume that h_0 does not belong to ΔX, then $\left|\theta^{\square}_X(h)\right|$ is a $(\mathrm{rk}(X)-1)$-surface. However, h_0 belongs to $\theta^{\square}_X(h)$, then $\left|\theta^{\square}_{\theta^{\square}_X(h)}(h_0)\right|$ is a $(\mathrm{rk}(X)-2)$-surface. We obtain a contradiction. Then $h_0 \in \Delta X$, $h_0 \in \theta^{\square}_X(h) \cap \Delta X$, that is, $h_0 \in \theta^{\square}_{\Delta X}(h)$. This concludes the proof. □

[1] A finite subset $X \subset \mathbb{Z}^n$ is said to be *digitally well-composed* (DWC) when c_{2n}- and c_{3^n-1}-connectivities are locally equivalent, that is, X does not contain any primary or secondary critical configuration (see [43] for more details).

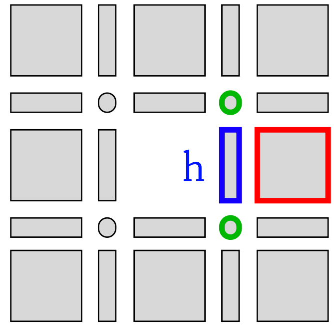


Fig. 13 The neighborhood in the border is not necessarily equal to the border of the neighborhood, even for n-PCMs. Here, we have a coherent 2-PCM depicted in gray with $\Delta X = X$, and $\theta^{\square}_{\Delta X}(h)$ which is equal to the set of gray faces contoured in green and red, when $\Delta\theta^{\square}_X(h)$ is equal to the set of the two gray vertices contoured in green.

The converse of Proposition 9 is not true (see Figure 13). Indeed, the depicted poset X is a 2-PCM since its is connected and for any $h \in X$, $\left|\theta^{\square}_X(h)\right|$ is a 1-PCM. Furthermore, $\Delta X = X$, which leads to $\Delta\theta^{\square}_X(h) \subsetneq \theta^{\square}_{\Delta X}(h)$ for any face h of X. This leads to the following remark.

Remark 11 Let X be a coherent poset of rank $n \geq 0$. The fact that X is an n-PCM does **not** imply that $\Delta\theta^{\square}_X(h)$ is equal to $\theta^{\square}_{\Delta X}(h)$ for any $h \in \Delta X$.

3.5 Topology of borders and interiors

Property 10 *Let Y be an n-surface with $n \geq 0$, and let X be a subset of Y of rank n. When X is closed in Y, then ΔX is closed in Y.*

Proof: Let a be an element of ΔX, which means that $|\theta^{\square}_X(a)|$ is not an $(n-1)$-surface. In other words, $|\beta^{\square}_X(a)|$ is not an $(n-\mathrm{rk}(a)-1)$-surface. Let b be an element of $\alpha^{\square}_X(a)$. We assume:

$$b \notin \Delta X, \quad (P)$$

then $|\theta^{\square}_X(b)|$ is an $(n-1)$-surface, and then $|\beta^{\square}_X(b)|$ is an $(n-1-\mathrm{rk}(b))$-surface. However, $a \in \beta^{\square}_X(b)$, which leads to the fact that $\left|\theta^{\square}_{\beta^{\square}_X(b)}(a)\right|$ is an $(n-$

rk$(b) - 2)$-surface. Also,

$$\begin{aligned}&\left|\theta^{\square}_{\beta^{\square}_X(b)}(a)\right|\\&=\left|\beta^{\square}_{\beta^{\square}_X(b)}(a)\right| * \left|\alpha^{\square}_{\beta^{\square}_X(b)}(a)\right|\\&=\left|\beta^{\square}_X(a)\cap\beta^{\square}_X(b)\right| * \left|\alpha^{\square}_X(a)\cap\beta^{\square}_X(b)\right|\\&=\left|\beta^{\square}_X(a)\right| * \left|\beta^{\square}_{\alpha^{\square}_X(a)}(b)\right|.\end{aligned}$$

Since $\left|\theta^{\square}_{\beta^{\square}_X(b)}(a)\right|$ is an $(n-\mathrm{rk}(b)-2)$-surface and $\left|\beta^{\square}_{\alpha^{\square}_X(a)}(b)\right|$ is a $(\mathrm{rk}(a)-\mathrm{rk}(b)-2)$-surface, we obtain by Proposition 2 that $\left|\beta^{\square}_X(a)\right|$ is a k-surface satisfying:

$$(n-\mathrm{rk}(b)-2)=k+(\mathrm{rk}(a)-\mathrm{rk}(b)-2)+1,$$

then $k=n-\mathrm{rk}(a)-1$. This contradicts (P), then $b\in\Delta X$, and then ΔX is closed in Y. ☐

Property 11 *Let Y be an n-surface with $n\geq 0$, and let X be a subset of Y of rank n. The set* $\mathrm{Int}(X)$ *is open in Y.*

Proof: Let h be an element of $\mathrm{Int}(X)$. By definition, $|\theta^{\square}_X(h)|$ is an $(n-1)$-surface since the rank of X is n. Because $X\subseteq Y$, $\theta^{\square}_X(h)\subseteq\theta^{\square}_Y(h)$. However, $|\theta^{\square}_X(h)|$ and $|\theta^{\square}_Y(h)|$ are nested $(n-1)$-surfaces, thus they are equal by Proposition 3. Because $|\theta^{\square}_X(h)|=|\theta^{\square}_Y(h)|$, $|\beta^{\square}_X(h)|=|\beta^{\square}_Y(h)|$. Since $\beta^{\square}_X(h)\subseteq X$, then $\beta^{\square}_Y(h)\subseteq X$. This last property being true for each $h\in\mathrm{Int}(X)$, $\mathrm{Int}(X)$ is an open set in Y. ☐

Remark: $\mathrm{Int}(X)$ is not always the greatest open set in Y contained in X. As counter-example, we can take the set X made of a 2-face h in $Y=\mathbb{H}^2$. At this moment, $\Delta X = X$, because $|\theta^{\square}_X(h)|$ is a (-1)-surface, and then not a 1-surface. Then, $\mathrm{Int}(X)=\emptyset$. However, the set X is open in $\mathbb{H}^2$ since $\beta_Y(h)=\{h\}\subseteq X$.

4 Introducing n-PCMs and smooth n-PCMs

Let us now introduce our definition of *poset-based connected n-manifolds* (shortly n-PCM), made possible thanks to the definition of border presented before.

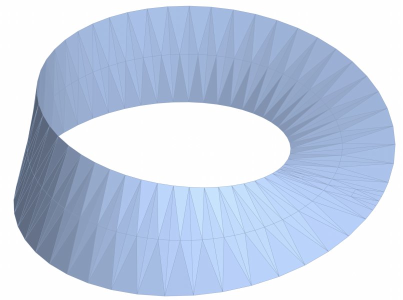


Fig. 14 An example of 2-PCM whose geometric realization draws a Möbius strip. Let us call X this poset, we can remark that for any face $h\in X$, either h belongs to $\mathrm{Int}(X)$ and $|\theta^{\square}_X(h)|$ is a 1-surface, or h belongs to ΔX and $|\theta^{\square}_X(h)|$ is a 1-PCM. Additionally, the border of X is a simple closed curve (a 1-surface), so X is also a smooth 2-PCM.

Definition 23 (n-PCM) Let X be a poset of rank $n\geq -1$. We say that X is a (-1)-PCM when it is the empty order. We say that X is a 0-PCM when X can be written $\{h\}$ with h an arbitrary element; and we say that X is an n-PCM, with $n\geq 1$, when X is connected, $\Delta X\neq\emptyset$, and for any $h\in X$, either $\left|\theta^{\square}_X(h)\right|$ is an $(n-1)$-surface (and h does not belong to ΔX), or it is an $(n-1)$-PCM (and h belongs to ΔX).

Please see Figure 14 for an example of 2-PCM.

Proposition 12 *The only poset being at the same time a PCM and a discrete surface is the empty order. In other words, this set is isomorphic to a -1-surface and to a -1-PCM.*

Proof: Let X be a poset which is for some $k\geq -1$ a k-surface and for some $k'\geq -1$ a k'-PCM. Since the border of any discrete surface is the empty set, we have the property: $\Delta X=\emptyset$. As a k'-PCM, $\Delta X=\emptyset$ implies that $k'\in\{-1,0\}$, so either X is a -1-PCM and it is the empty set, or it contains one element and it is a 0-PCM. It cannot be a 1-PCM since no discrete surface is made of only one element (either a discrete surface is empty, or it contains at least two elements). Consequently, X is a -1-PCM, and it is the empty order. The proof is done. ☐

Remark 12 When X is an n-PCM with $n\geq 1$, then ΔX is the set of elements h of X satisfying that $\left|\theta^{\square}_X(h)\right|$ is an $(n-1)$-PCM.

Proposition 13 *A discrete n-surface and an n-PCM contain at least $n+1$ elements.*

Proof: Let us treat the discrete surface case. We start from the case $k=-1$ which is naturally satisfied. Then, we can prove by induction that if a k-surface contains at least $k+1$ elements, a $k+1$-surface contains at least $k+2$ elements: let X be a $k+1$-surface, then for any $h \in X$, $|\theta^{\square}_X(h)|$ is a k-surface and contains at least $k+1$ elements. Since X contains $\{h\}$ and $\theta^{\square}_X(h)$ (two terms whose union is disjoint), X contains at least $k+1$ elements. The proof is done for the discrete surfaces. Concerning the PCM's, the reasoning is the same. The proof is done. □

Proposition 14 ($n=0$) *When X is a poset of rank* 0*, then X is a* 0*-PCM when its cardinality is equal to* 1*, a* 0*-surface when its cardinality is equal to* 2*, and neither a discrete surface nor a PCM when its cardinality is greater than* 2*.*

Proof: It is the direct consequence of the definitions of what are 0-surfaces and 0-PCM's. □

Proposition 15 ($n=1$) *A simple open path is a* 1*-PCM and conversely. Furthermore, a simple closed curve is a* 1*-surface and conversely.*

Proof: Let us treat the case of 1-PCM's.

Let π be some simple open path, then it is connected (by construction), and each end point $e_1, e_2 \in \pi$ satisfies that $|\theta^{\square}_{\pi}(e_i)|$ is a 0-PCM for $i \in \{1,2\}$. Also, any element h of $\pi \setminus \{e_1, e_2\}$ satisfies that $|\theta^{\square}_{\pi}(h)|$ is a 0-surface. So, a simple open path is a 1-PCM.

Let X be a 1-PCM. It is connected by definition. Moreover, $\Delta X \neq \emptyset$. Let h_0 be an element of ΔX. By definition of a 1-PCM, $\theta^{\square}_X(h_0)$ is made of one element that we call h_1. We also have that $|\theta^{\square}_X(h_1)|$ is either a 1-PCM (in this case, X is made of two elements and draws a simple open path) or a 0-surface. When $|\theta^{\square}_X(h_1)|$ is a 0-surface, since $h_0 \in \theta^{\square}_X(h_1)$, there exists one and only one other element h_2 of X which belongs to $\theta^{\square}_X(h_1)$, and h_0 and h_2 are not neighbors. If $|\theta^{\square}_X(h_2)|$ is a 0-PCM, we reached the other end point h_2 of $X=\{h_0,h_1,h_2\}$ which is then a simple open path. If $|\theta^{\square}_X(h_2)|$ is a 0-surface, we can re-iterate as before until we reach the other end point of X. This point exists since X is finite. Moreover, X is simple as a path since no self-crossing can occur: for any $h \in X$, the cardinality of $|\theta^{\square}_X(h)|$ is lower than or equal to two. Last, X is open as a path since it owns two end points. We have then proved that X is a simple open path.

Let us now treat the case of discrete 1-surfaces.

Let π_0 be a simple closed curve, then for any $h \in \pi_0$, we have $\theta^{\square}_{\pi_0}(h)$ is made of two elements of π_0 which are not neighbors, so $|\theta^{\square}_{\pi_0}(h)|$ is a 0-surface. Furthermore, π_0 is connected, so it is a 1-surface.

When X is a 1-surface, we start from any element of $h_0 \in X$. Since $|\theta^{\square}_X(h_0)|$ is a 0-surface, it contains two elements h_{-1} and h_1 which are not neighbors. Thus, either $h_{-1}, h_1 \in \alpha_X(h_0)$, or $h_{-1}, h_1 \in \beta_X(h_0)$. We consider the first case, since the reasoning for the other case is the same. We know that $|\theta^{\square}_X(h_1)|$ is a 0-surface and that $h_0 \in \beta^{\square}_x(h_1)$, thus h_2 belongs to $\beta^{\square}_x(h_1)$. We can remark that h_{-1} is different from h_2: if $h_{-1}=h_2$, then $h_2 \in \alpha^{\square}_X(h_0)$ so they are neighbors and we obtain a contradiction. So, $h_{-1} \neq h_2$. Now, two cases are possible: either the 0-surface $|\theta^{\square}_X(h_2)|$ contains h_{-1} and we close the loop (we obtain a simple closed curve), or it does not contain h_{-1} and there exists h_3 in $\theta^{\square}_X(h_2)$ then equal to $\{h_1, h_3\}$. We continue this "propagation" until we reach h_{-1}, which necessarily happens since X is finite by hypothesis, leading to a closed curve. This closed curve is the only component of X since X is connected by hypothesis. Furthermore, X is simple as a closed curve since any self-crossing at $h \in X$ would lead to a term $\theta^{\square}_X(h)$ containing more than two elements, which is impossible since X is a 1-surface. The proof is done. □

Remark 13 Let Y be a poset. We call the poset $|X| := |\{a,b\}|$ with a and b neighbors in Y a *degenerated* 1*-PCM*. This poset satisfies the relation $\Delta X = X$.

Proposition 16 (Elementary property of the border) *The border of a* 0*-surface or of a* 0*-PCM is an empty set.*

Proof: Let $|X| = |\{a,b\}|$ be a 0-surface, with $a \notin \theta_X(b)$. Then, for any $h \in X$, $|\theta^{\square}_X(h)| = |\emptyset|$

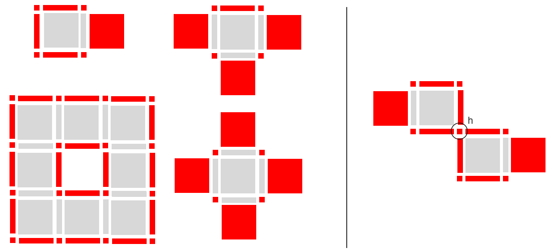


Fig. 15 Different posets whose interiors are drawn in gray, and whose borders are drawn in red. On the left side, several $\mathcal{P}$-well-composed 2-PCM's. On the right side, a 2-PCM X which is not $\mathcal{P}$-well-composed: as we can see, the neighborhood of h in the border is not a 0-surface (four 1-faces), therefore the border ΔX is not a 1-surface, thus X is not $\mathcal{P}$-well-composed.

which is a (-1)-surface, which means that h does not belong to ΔX. Then, $\Delta X = \emptyset$. When $|X| = \{a\}$ is a 0-PCM, then $|\theta_X^\square(a)| = |\emptyset|$ and then $a \notin \Delta X$. Then $\Delta X = \emptyset$ too. $\square$

Definition 24 (Smooth PCMs) Let X be a poset of rank $n \geq -1$. We say that X is a *smooth* -1*-PCM* when $X = \emptyset$, that X is a *smooth* 0*-PCM* when $X = \{h\}$ with h some arbitrary element, and that X is a *smooth* n*-PCM*, $n \geq 1$, when it is connected, its border $\Delta X \neq \emptyset$ is an $(n-1)$-surface or a separated union of $(n-1)$-surfaces, and for any $h \in X$, $|\theta_X^\square(h)|$ is either a smooth $(n-1)$-PCM (border case) or an $(n-1)$-surface (interior case).

In Figure 14, we can see an example of smooth 2-PCM. However, there exist PCM's which are not smooth: we can observe the example showing a 1-PCM which is not a smooth 1-PCM is detailed in Remark 13. Note that this non-equivalence can be generalized to dimension $n \geq 1$ (see Figure 13 for the case $n = 2$). This leads to the following remark.

Remark 14 A smooth n-PCM of rank $n \geq 2$ is $\mathcal{P}$-well-composed, but a $\mathcal{P}$-well-composed poset of rank $n \geq 2$ is not necessarily a smooth n-PCM (since X is not necessarily connected). The question whether a $\mathcal{P}$-well-composed connected poset is a smooth n-PCM when $n \geq 2$ remains open.

Remark 15 A smooth n-PCM is always an n-PCM, however an n-PCM is not always a smooth n-PCM: Figure 13 shows a 2-PCM which is not a smooth 2-PCM. In other words, a n-PCM is not always $\mathcal{P}$-well-composed.

Remark 16 Let X be a smooth n-PCM with $n \geq -1$, then:

$$\Delta\Delta X = \emptyset.$$

Proposition 17 *Let X be a smooth n-PCM, then X is coherent.*

Proof: When $n = -1$, X is the empty order, then it is coherent. When $n \geq 0$, we assume that when X' is a smooth $(n-1)$-PCM, it implies that X' is coherent (induction hypothesis). Let us show that when X is a smooth n-PCM, $n \geq 0$, it is coherent.

Let X be a smooth n-PCM, $n \geq 0$. For any $h \in X$, either $h \in \Delta X$ and $|\theta_X^\square(h)|$ is a smooth $(n-1)$-PCM, thus coherent and of rank $(n-1)$ (by the induction hypothesis), or $h \in \mathrm{Int}(X)$, and $|\theta_X^\square(h)|$ is an $(n-1)$-surface and thus coherent and of rank $(n-1)$ too. In other words, for any $h \in X$, $|\theta_X^\square(h)|$ is coherent of rank $(n-1)$, thus X is coherent. $\square$

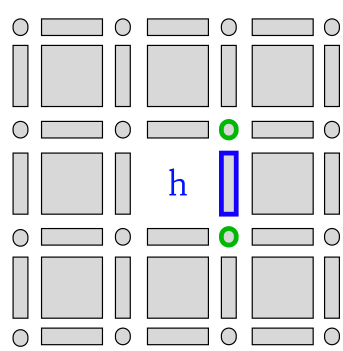


Fig. 16 In a smooth n-PCM, the neighborhood of the border is equal to the border of the neighborhood. Starting from h belonging to the border ΔX of X, it leads to the same results when we compute first the neighborhood in X of h and then its border, or if we compute first the border and then the neighborhood of h in ΔX. Intuitively, this nice property comes from the fact that the border of this smooth 2-PCM is made of **separated** 1-surfaces.

Proposition 18 *When X is a smooth n-PCM with $n \geq 1$, for any $h \in \Delta X$, we have the relation:*

$$\theta^{\square}_{\Delta X}(h) = \Delta\theta^{\square}_{X}(h).$$

*In other words, in smooth n-PCM's, the neighborhood of the border and the border of the neighborhood are **equal**.*

Proof: This property is illustrated in Figure 16.

When $n = 1$, the equality is obvious. Let us treat the case $n \geq 2$.

When X is a smooth n-PCM with $n \geq 2$, it follows that ΔX is a separated union of $(n-1)$-surfaces, and also that for any $h \in \Delta X$, $\Delta\theta^{\square}_{X}(h)$ is a separated union of $(n-2)$-surfaces (since $|\theta^{\square}_{X}(h)|$ is a smooth $(n-1)$-PCM).

By Proposition 9, $\Delta\theta^{\square}_{X}(h) \subseteq \theta^{\square}_{\Delta X}(h)$. Furthermore, ΔX is a separated union of $(n-1)$-surfaces, thus $|\theta^{\square}_{\Delta X}(h)|$ is an $(n-2)$-surface. Besides, $\Delta\theta^{\square}_{X}(h)$ is a separated union of $(n-2)$-surfaces.

When $n = 2$, $|\theta^{\square}_{\Delta X}(h)|$ is a 0-surface and $\Delta\theta^{\square}_{X}(h)$ is made of at least one 0-surface. Thanks to the inclusion $\Delta\theta^{\square}_{X}(h) \subseteq \theta_{\Delta X}(h)$, it follows that $\Delta\theta^{\square}_{X}(h)$ is one 0-surface and it is equal to $\theta^{\square}_{\Delta X}(h)$.

When $n \geq 3$, each connected component of $\Delta\theta^{\square}_{X}(h)$ is an $(n-2)$-surface contained in the $(n-2)$-surface $|\theta^{\square}_{\Delta X}(h)|$. Thus, by Proposition 3, $\theta^{\square}_{\Delta X}(h) = \Delta\theta^{\square}_{X}(h)$. □

Proposition 19 *Let X be a smooth n-PCM, $n \geq 0$. For any $h \in \Delta X$, we have the relation:*

$$\mathrm{Int}(\theta^{\square}_{X}(h)) \subseteq \mathrm{Int}(X).$$

Proof: Let h be an element of ΔX, and h' be an element of $\mathrm{Int}(\theta^{\square}_{X}(h)) \subseteq \theta^{\square}_{X}(h)$. It leads to $h' \in \theta^{\square}_{X}(h)$ and $h' \notin \Delta\theta^{\square}_{X}(h)$.

This is equivalent to say that $h' \in \theta^{\square}_{X}(h)$ and $h' \notin \theta^{\square}_{\Delta X}(h)$ by Proposition 18. Thus, $h' \in \theta^{\square}_{X}(h)$ and $h' \notin \theta^{\square}_{X}(h) \cap \Delta X$, that is, $h' \notin \Delta X$.

However, $h' \in \mathrm{Int}(\theta^{\square}_{X}(h))$ implies that h' belongs to $\theta^{\square}_{X}(h)$ and then to X. Thus, h' belongs to $\mathrm{Int}(X)$. □

5 Our main theorems

In the sequel, we present fundamental theorems, showing that depending on the hypotheses, a union of two smooth n-PCM's can lead to a discrete n-surface or to a n-PCM. We also show that cutting a discrete n-surface leads to one or two smooth n-PCM's. All the proofs are provided in Section A.

5.1 Unions of PCM's

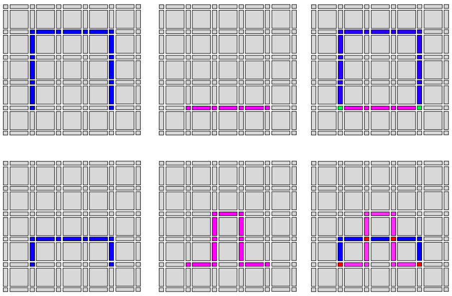


Fig. 17 Intuition of Theorem 20. In the 1D case, starting from two simple closed paths π_1 and π_2 which satisfy (see the top row made of three cubical complexes): (1) the interior of π_1 (that is, π_1 minus its extremities) is not neighbour of the interior of π_2, and (2) they have the same start and end points, then we are ensured that the union $\pi_1 \cup \pi_2$ is a simple closed curve. Indeed, we closed the curve in a way that the result is simple (no self-crossing). A counter-example is depicted on the bottom row made of the three other cubical complexes: the intersection of the interiors lead to a poset which is not a simple closed curve (due to the two self-crossings). Now, in n-D, the concept is exactly the same: we need that the interior of each n-PCM is not neighbor of the interior of the other n-PCM, and that they share their boundaries. Under these constraints, their union is a discrete n-surface.

Remark 17 Let A, B be two posets subsets of a poset X. The relation $\theta_X(\mathrm{Int}(A)) \cap \mathrm{Int}(B) = \emptyset$ is symmetrical, that is, $\{\theta_X(\mathrm{Int}(A)) \cap \mathrm{Int}(B) = \emptyset\}$ is equivalent to $\{\mathrm{Int}(A) \cap \theta_X(\mathrm{Int}(B)) = \emptyset\}$. This symmetry is a direct consequence of the symmetry of the operator θ.

Theorem 20 *Let A, B be two smooth n-PCMs, with $n \geq 0$, such that $\mathrm{rk}(A \cup B) = n$, $\Delta A = \Delta B = A \cap B$, and satisfying $\theta(\mathrm{Int}(A)) \cap \mathrm{Int}(B) = \emptyset$. Then, $A \cup B$ is an n-surface.*

The intuition of this theorem is given in Figure 17.

Theorem 21 *Let A, B be two smooth n-PCMs with $n \geq 1$. When $A \cap B = \Delta A \cap \Delta B$ is a smooth $(n-1)$-PCM, the rank of $A \cup B$ is equal to n, $\theta(A \setminus B) \cap (B \setminus A) = \emptyset$, and $\theta(\mathrm{Int}(A)) \cap \mathrm{Int}(B) = \emptyset$, then $A \cup B$ is an n-PCM, and its border is equal to:*

$$\Delta(A \cup B) = (\Delta A \setminus B) \cup \Delta(A \cap B) \cup (\Delta B \setminus A).$$

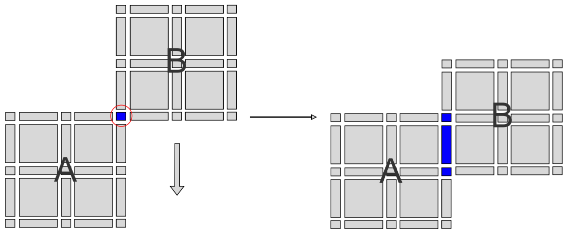


Fig. 18 Intuition of Theorem 21. If we group together two 2D cubical complexes A and B, both assumed to be smooth 2-PCM's, so that they only share a vertex, we surely form a "pinch" that we will call h. At this pinch, the neighborhood in $A \cup B$ is not connected, so $\theta^{\square}_{A \cup B}(h)$ is neither a 1-PCM nor a 1-surface. So, $A \cup B$ is not a PCM. Now, if we move a little the set B so that A and B share a 1-PCM, we see that there is no pinch anymore, and the union of A and B is a 2-PCM.

An intuition of Theorem 21 is given in Figure 18.

Note that proving that $\Delta(A \cup B)$ in the previous theorem is an $(n-1)$-surface or a separated union of $(n-1)$-surfaces, which could allow us to conclude that the result is not only an n-PCM but also a smooth n-PCM is an open problem. Indeed, it means that we need to prove that a union of three posets with no particular topological properties can be merged in a way that we obtain a discrete $(n-1)$-surface (when all our previous proofs refer only to pairs of posets).

5.2 Decomposition of n-surfaces into smooth PCMs

To complete the previous theorem, we can show that when we "cut" a discrete surface in a nice way, we can obtain smooth n-PCMs.

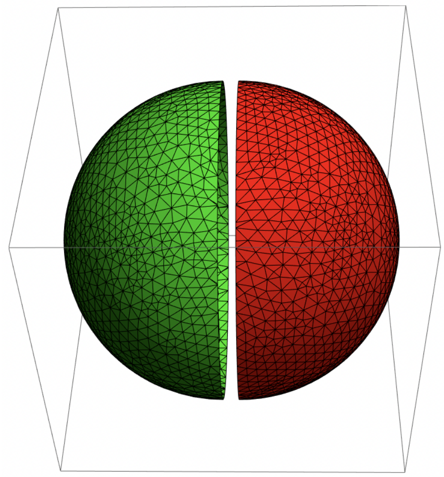


Fig. 19 Intuition of Theorem 22. Starting from a triangulated 2-sphere, we obtain a discrete 2-surface. Now, we cut this sphere into two pieces following a simple closed curve N drawn on this surface. We obtain two halves of spheres whose boundaries is this same N. Since PCM's and discrete surfaces satisfy the same properties (except at the border), we surely obtain two 2-PCM's : the red and green pieces. Since the boundary N of each piece is a 1-surface (by hypothesis), these two pieces are smooth PCM's.

Theorem 22 *Let X be an n-surface with $n \geq 1$, and let N be some suborder of X which is an $(n-1)$-surface. We denote then by $\{C_i\}_{i \in \mathcal{I}}$ the connected components of $\mathcal{CC}(X \setminus N)$. We assume that $\#(\mathcal{I}) \geq 2$. Then, the components $|C_i \sqcup N|$ are smooth n-PCMs.*

An intuitive explanation of this theorem is provided in Figure 19.

Note: The case $\#(\mathcal{I}) \geq 2$ in the previous theorem is to avoid the case where the $(n-1)$-face cuts a "handle" of the given n-surface, that is, does not separate the initial n-surface into several components. Indeed, in the 2-torus case, the component $|C_0 \sqcup N|$ is equal to the initial n-surface.

5.3 Cardinality of cuts of discrete surfaces

Theorem 23 *Let $n \geq 1$ be some integer. Let X be a discrete n-surface and let $N \subset X$ be a discrete $(n-1)$-surface. Let us denote $\{C_i\}_{i \in \mathcal{I}} = \mathcal{CC}(X \setminus N)$. Then, the cardinality of $\mathcal{I}$ is equal to or lower than two. In other words, the "cut" of a discrete surface using a discrete surface of lower rank leads to at most two smooth n-PCM's.*

The intuition behind this theorem is simple for the case $n = 2$: if we start from some discrete 2-surface like a sphere, and that we "cut" it along an 1-surface N, we will obtain two 2-PCM's whose boundary equals N. Starting from a torus, depending on how we cut it, we will obtain either one or two pieces with the same boundary N. In the more general case and for any $n \geq 2$, when we assume that the cut led to three pieces or more, we know that the boundaries of all these pieces is the same $(n-1)$-surface N. By Theorem 22, we also know that we can group two of these pieces to form an n-surface. This means that the n-surface made of the two pieces in contained in the union of all the pieces, which is also an n-surface. Then, they are equal. This leads to the contradiction that we obtain three pieces at least after the cut. So, each cut leads to one or two pieces.

6 Joint of PCMs and n-surfaces

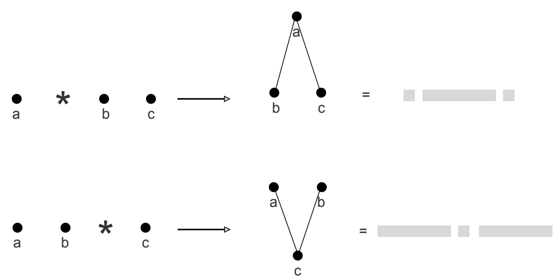


Fig. 20 The join of one 0-surface $\mathcal{S}_0$ with a 0-PCM $\mathcal{B}_0$ leads to a 1-PCM. The same applies for the join of $\mathcal{B}_0$ and $\mathcal{S}_0$, even if the result is not the same (in fact, it is its dual).

Now, let us present some intuitive propositions relative to the order joint of PCMs and surfaces (see Figure 20 for some intuitive examples).

We recall that the mathematical background relative to joins is placed at the end of Section 2.1.

Proposition 24 *Let k, ℓ be two non negative integers. The order joint of a k-surface $\mathcal{S}_k$ and of a ℓ-PCM $\mathcal{B}_\ell$ is a $(k+\ell+1)$-PCM. Furthermore, $\Delta(\mathcal{S}_k * \mathcal{B}_\ell) = \Delta\mathcal{B}_\ell \cup \mathcal{S}_k$.*

Proof: We assume that k and ℓ are non-negative integers. Let us proceed to a proof by induction on $(k+\ell)$. We denote by $\mathcal{S}_k$ a k-surface and by $\mathcal{B}_\ell$ a ℓ-PCM.

Initialization $(k+\ell=0)$: We have $k=\ell=0$, then $\mathcal{S}_0 = |\{a,b\}|$ with a and b that are not neighbors and $\mathcal{B}_0 = |\{c\}|$. The joint $|\mathcal{S}_0 * \mathcal{B}_0|$ is equal to $|\{a,b\}| * |\{c\}| = |\{(c,a),(c,b)\}|$ (where (x,y) means that x is in the closure of y). The border of $|\mathcal{S}_0 * \mathcal{B}_0|$ is $\{a,b\}$, and when h belongs to it, $\left|\theta^{\square}_{\mathcal{S}_0 * \mathcal{B}_0}(h)\right| = \left|\theta^{\square}_{\mathcal{S}_0}(h) * \mathcal{B}_0\right| = |\mathcal{B}_0|$ which is a 0-PCM. When h belongs to $\mathcal{S}_0 * \mathcal{B}_0$ minus its border, then $h = c$ and $\left|\theta^{\square}_{\mathcal{S}_0 * \mathcal{B}_0}(h)\right| = |\{a,b\}|$ which is a 0-surface. Since a joint of two non-empty posets is connected, $|\mathcal{S}_0| * |\mathcal{B}_0|$ is a 1-PCM and the property is proved for $k=\ell=0$.

Heredity $(k+\ell \geq 1)$: We assume that the property is true for $k'+\ell' \in [0, k+\ell-1]$ and we want to prove it for $(k+\ell)$. Let $\mathcal{S}_k$ be a k-surface, and $\mathcal{B}_\ell$ be a ℓ-PCM. When $h \in \mathcal{S}_k * \mathcal{B}_\ell$, three cases are possible:

- When $h \in \mathcal{S}_k$, then $\left|\theta^{\square}_{\mathcal{S}_k * \mathcal{B}_\ell}(h)\right| = \left|\theta^{\square}_{\mathcal{S}_k}(h)\right| * |\mathcal{B}_\ell|$ which is the joint of a $(k-1)$-surface and of a ℓ-PCM, that is, a $(k+\ell)$-PCM thanks to the induction hypothesis.
- When $h \in \Delta\mathcal{B}_\ell$, then $\left|\theta^{\square}_{\mathcal{S}_k * \mathcal{B}_\ell}(h)\right| = |\mathcal{S}_k| * \left|\theta^{\square}_{\mathcal{B}_\ell}(h)\right|$ which is the joint of a k-surface and of a $(\ell-1)$-PCM, that is, a $(k+\ell)$-PCM thanks to the induction hypothesis.
- When $h \in \mathcal{B}_\ell \setminus \Delta\mathcal{B}_\ell$, then $\left|\theta^{\square}_{\mathcal{S}_k * \mathcal{B}_\ell}(h)\right| = |\mathcal{S}_k| * \left|\theta^{\square}_{\mathcal{B}_\ell}(h)\right|$ which is the joint of a k-surface and of a $(\ell-1)$-surface, that is, a $(k+\ell)$-surface.

In other words, $\Delta(\mathcal{S}_k * \mathcal{B}_\ell) = \Delta\mathcal{B}_\ell \cup \mathcal{S}_k$ and since $\mathcal{S}_k * \mathcal{B}_\ell$ is the joint of two non-empty posets, it is connected and it is then a $(k+\ell+1)$-PCM. □

Proposition 25 *The order joint of a k-PCM $\mathcal{B}_k$ and of a ℓ-surface $\mathcal{S}_\ell$ is a $(k+\ell+1)$-PCM. Furthermore, $\Delta(\mathcal{B}_k * \mathcal{S}_\ell) = \mathcal{S}_\ell \cup \Delta\mathcal{B}_k$.*

Proof: This proof is similar to the previous one and is left to the reader. □

7 Applications

We start by presenting some applications of PWCness related to the tree of shapes and obtained thanks to the definition of borders. Then, we follow with some nice properties of n-PCM's.

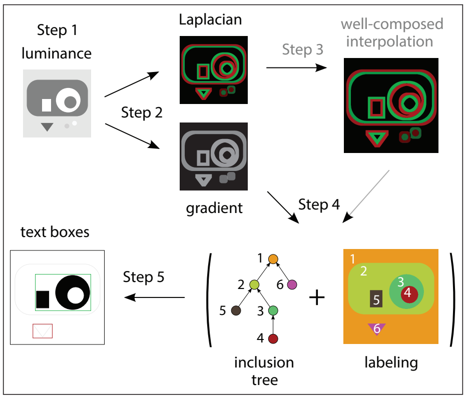
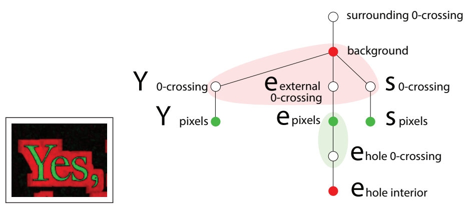


Fig. 21 Summary of the method used by Huyhn *et al.* [59]



(a) $\Delta^{wc}(u)$. (b) Tree $\mathfrak{S}(\operatorname{sign}(\Delta^{wc}(u)))$.

Fig. 22 An inclusion tree and its corresponding image [59]

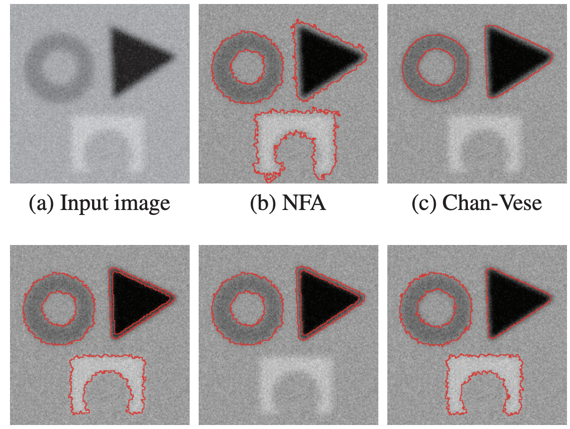
7.1 Some applications of the tree of shapes

Let U be some $\mathcal{P}$-well-composed plain map defined on a smooth n-PCM X : $U : X \rightsquigarrow \mathbb{Z}$. Let us assume that we can embed the poset X into a greater space X_0 of rank n, with X_0 an n-surface. By Proposition 5, we know that PWCness and AWCness are equivalent in this context, so U is AWC too, and its tree of shapes is well-defined. Consequently, all the well-known application of the tree of shapes become accessible. Let us describe some of them.

7.1.1 Application to the zero-crossing of the Laplacian

We present some results of Huyhn *et al.* [59] obtained thanks to the computation of the tree of the sign of the AWC morphological Laplacian in a self-dual way. It is used here for text detection, but this approach can easily be extended to treat n-D signals, such as MR images, videos, or CT-scans.

This approach is part of the connected-components-based ones and consists in transforming an image into a tree-based hierarchical representation (see Figure 21), based on adjacency and inclusion relationship between the components in the image. To proceed, they compute the Laplacian of a given image using a *morphological Laplacian operator* [60], whose *zero-crossings* are known to be very precise contour estimations of the initial image. After that, a self-dual well-composed interpolation [46] of this Laplacian is computed; this way, the zero-crossings of this interpolation are simple closed curves. Using these separated Jordan curves, we can naturally induce a hierarchy [45] in the image: saturation of these curves (whatever the chosen connectivity) are either nested or disjoint. A component labeling of the sign of the Laplacian and the generation of the inclusion tree are then straightforward and very fast. Thanks to this representation, they can extract text candidates: a hole of a character or a solid character are leafs of the tree (see Figure 22), and so on. Text grouping is then a subtree, since characters must be grouped iff they belong to the same background. Finally, in this context, well-composedness gave access to a very fast (linear time) and efficient self-dual text detection algorithm thanks to the hierarchy induced by the Jordan curves extracted from the well-composed Laplacian.



(a) Input image (b) NFA (c) Chan-Vese

(d) Ballester, $\lambda = 2k$ (e) Ballester, $\lambda = 3k$ (f) Our method

Fig. 23 Segmentation results on synthetic data. This figure has been extracted from [35].

7.1.2 Energy minimisation using the tree of shapes

In [35], the authors propose a framework based on energy minimisation relative to the context (using the tree of shapes) to proceed to segmentation.

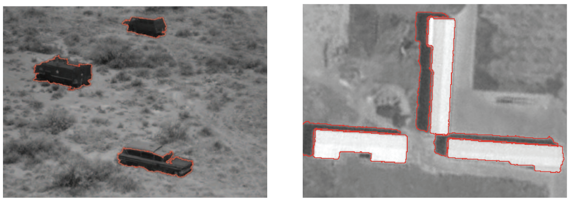


Fig. 24 Segmentation results on real data. This figure has been extracted from [35].

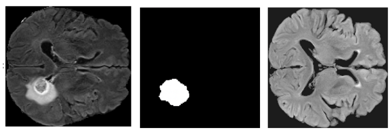


Fig. 25 From left to right, a T2 MR brain image containing a tumor, the ground truth locating the tumor in the image space, and the sane brain which matches the most to the initial tumored brain. This figure has been extracted from [38].

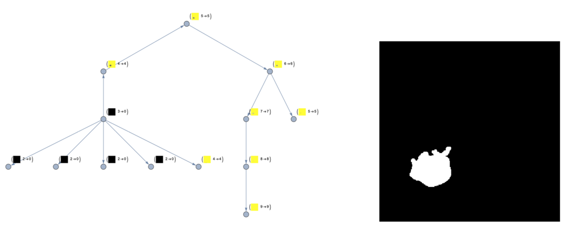


Fig. 26 The subtree corresponding to the tumor and the reconstructed tumor area. This figure has been extracted from [38].

Some results can be observed in Figures 23 and 24 and show the robustness of the approach to noise. Thanks to the tree structure, the estimator can be computed incrementally in an efficient fashion. Experimental results on synthetic and real images demonstrate the robustness and usefulness of their method.

7.1.3 Biomedical image segmentation

We recall first that DWCness and PWCness are equivalent on cubical grids (see Proposition 8). In [38], they propose to start from an image of a tumored brain coming from some dataset. Then, they look for the sane brain which matches the most in another data set (see Figure 25). They compute the tree of shapes of these two images made DWC using some DWC interpolation [45, 46]. By computing the *differentiation* of these two trees, they are able to extract the subtree (see Figure 26) of the second tree of shapes, which represents the hierarchical representation of the tumor in the image space. This results then in a hierarchical segmentation and not only the tumor boundary, what represents the originality of this approach.

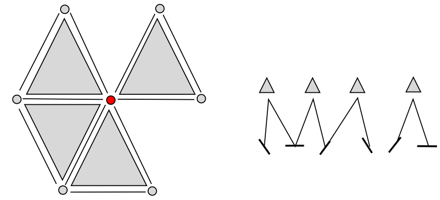


Fig. 27 A pinch of rank 0 in a poset X of rank 2. We can see that the face $h \in \Delta X$ satisfies the property that $|\theta_X^\square(h)|$ is disconnected (see its Hasse diagram on the right side), so it is not a (smooth) 1-PCM. Thus h is a pinch (of rank 0).

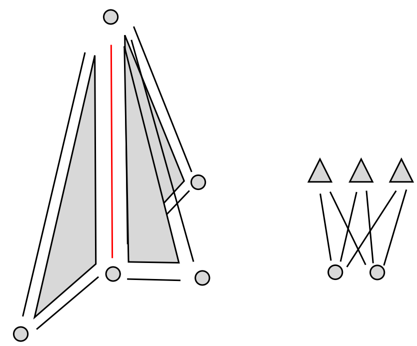


Fig. 28 A pinch of rank 1 in a poset X of rank 2. We can see that the face $h \in \Delta X$ satisfies the property that $|\theta_X^\square(h)|$, even if connected, is not a 1-PCM since it is not an open simple path (see its Hasse diagram on the right side). Since $|\theta_X^\square(h)|$ is not a 1-PCM, it is not a smooth 1-PCM neither, so h is a pinch (of rank 1).

7.2 Application in digital geometry

Checking whether a simplicial manifold (like a triangulation) is a combinatorial manifold is particularly hard, since we have to check if the links of the vertices are spheres or balls. However, in the case of posets, which is an even more general case, we are going to show that there exists a very simple, tractable, and recursive manner, using n-PCM's, to detect pinches. That is why we think that n-PCM's can be useful in digital geometry [61].

Definition 25 (Pinch) Let X be some poset of rank $n \geq 0$. If any face h of ΔX satisfies the property:

$$\left\{|\theta_X^{\Box}(h)| \text{ is not a smooth } (n-1)\text{-PCM}\right\},$$

we say that there exists a *pinch* located at h in X. If h is a k-face of X, we say that h is a *pinch of rank k* in X.

We depict in Figures 27 and 28 two pinches, of rank 1 and of rank 2 respectively. This is typically these configurations that we do want in discrete topology (since these pinches can lead to topological issues like sets with one exterior component but with two interior components), or in digital geometry (see for example the Marching cubes [62] where pinches lead to ambiguities [43]) .

8 Conclusion

In this paper, we have introduced a new definition of border which allows us to define poset-based connected manifolds, the counterpart of topological manifolds with borders in the discrete settings. These PCMs have several strong properties: under some constraints, they can be "glued" to make discrete surfaces, they can result from the cut of a discrete n-surface, we can glue two smooth n-PCMs to obtain an n-PCM, and n-PCMs and discrete surfaces can be joined to make a PCM of higher dimension. These properties are very desirable in discrete topology and in digital geometry.

Still thanks to this definition of border, we have provided a new flavour of well-composedness which does not depend on the boundary but on the border, that is, a poset does not need to lie in a greater ambient space to be well-composed. We have also shown that $\mathcal{P}$-well-composedness is compatible with well-composedness in the sense of Alexandrov in the sense that when a unicoherent poset lies into a greater ambient space, all the applications available for AWC sets work for PWC sets.

As future works, we plan to solve the open problem detailed above (that is, under which conditions the union of two smooth PCM's lead to a smooth PCM), and to generalize the definition of n-PCMs to more complex structures.

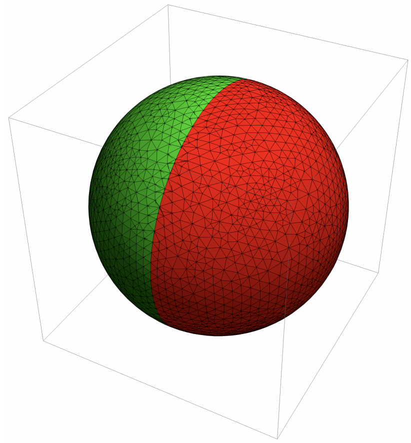


Fig. A1 The union of two smooth 2-PCM's makes an 2-surface under some conditions: the green poset A and the red poset B are both smooth 2-PCM's, and they share a 1-surface (their respective borders). We can observe that their union is indeed a 2-surface.

Appendix A Proofs of the main theorems

A.1 Unions of PCM's

Theorem 20. *Let A, B be two smooth n-PCMs, with $n \geq 0$, such that $\mathrm{rk}(A \cup B) = n$, $\Delta A = \Delta B = A \cap B$, and satisfying $\theta(\mathrm{Int}(A)) \cap \mathrm{Int}(B) = \emptyset$. Then, $A \cup B$ is an n-surface.*

Proof: We want to show that $X := A \cup B$ is an n-surface (see Figure A1). For this aim, we proceed by induction.

Initialization ($n = 0$): the case $n = 0$ is obvious since X is a set of two arbitrary elements which are not neighbors.

Heredity ($n \geq 1$): We can assume that when A' and B' are two smooth $(n-1)$-PCMs with $A' \cup B'$ or rank $(n-1)$, and the relations:

$$\theta(\mathrm{Int}(A')) \cap \mathrm{Int}(B') = \emptyset,$$

and

$$\Delta A' = \Delta B' = A' \cup B',$$

then $A' \cup B'$ is an $(n-1)$-surface (induction hypothesis).

Now, let us assume that we have A, B two smooth n-PCMs with the properties described above:

- Since $A \cap B = \Delta A \neq \emptyset$, $A \cup B$ is connected.

- For any $h \in A \setminus B = \text{Int}(A)$,
$$\theta^{\square}_{A\cup B}(h) = \theta^{\square}_{A}(h) \cup (\theta^{\square}(h) \cap B \setminus A).$$
However, $(\theta^{\square}(h) \cap B \setminus A) = \emptyset$ since $\theta(\text{Int}(A)) \cap \text{Int}(B) = \emptyset$, thus $\theta^{\square}_{A\cup B}(h) = \theta^{\square}_{A}(h))$, which is an $(n-1)$-surface.
- For any $h \in B \setminus A$, the symmetrical reasoning applies.
- When h belongs to $A \cap B$,
$$\begin{aligned}\theta^{\square}_{A\cup B}(h) &= \theta^{\square}(h) \cap (A \cup B)\\ &= \theta^{\square}_{A}(h) \cup \theta^{\square}_{B}(h)\\ &= A' \cup B',\end{aligned}$$
when we denote $\theta^{\square}_{A}(h)$ by A' and $\theta^{\square}_{B}(h)$ by B'. However:
  - A' is a smooth $(n-1)$-PCM since $h \in \Delta A$,
  - B' is a smooth $(n-1)$-PCM since $h \in \Delta B$,
  - $A' \cup B' = \theta^{\square}_{A\cup B}(h)$ is at most of rank $(n-1)$ since $A \cup B$ is of rank n by hypothesis, it is at least of rank $(n-1)$ since it contains $\theta^{\square}_{A}(h)$ which is of rank $(n-1)$, thus $A' \cup B'$ is of rank $(n-1)$,
  - thanks to Proposition 19:
  $$\text{Int}(A') \cap \theta(\text{Int}(B')) \subseteq \text{Int}(A) \cap \theta(\text{Int}(B)) = \emptyset,$$
  - thanks to Proposition 18, $\Delta A' = \theta^{\square}_{\Delta A}(h) = \theta^{\square}_{\Delta B}(h) = \Delta B'$, and $\theta^{\square}_{\Delta A}(h) = \theta^{\square}_{A\cap B}(h) = A' \cap B'$, thus $\Delta A' = \Delta B' = A' \cap B'$.

Thus, $A \cup B$ is an n-surface.

This concludes the proof. □

A.2 From unions of PCM's to PCM's

Theorem 21. *Let A, B be two smooth n-PCMs with $n \geq 1$. When $A \cap B = \Delta A \cap \Delta B$ is a smooth $(n-1)$-PCM, the rank of $A \cup B$ is equal to n, $\theta(A \setminus B) \cap (B \setminus A) = \emptyset$, and $\theta(\text{Int}(A)) \cap \text{Int}(B) = \emptyset$, then $A \cup B$ is an n-PCM, and its border is equal to:*
$$\Delta(A \cup B) = (\Delta A \setminus B) \cup \Delta(A \cap B) \cup (\Delta B \setminus A).$$

Proof: Let us proceed by induction on n.

Initialization ($n = 1$): Let A, B two smooth 1-PCMs. Since $A \cap B$ is a smooth 0-PCM by hypothesis, it is not empty, thus $A \cup B$ is connected. Furthermore, we can assume that A is an open simple path from a to h, and B is an open simple path from h to b, with a, h, b three arbitrary elements such that $a \notin \theta(b)$. We have then the following cases:

- When $z = a$,
$$\theta^{\square}_{A\cup B}(z) = (\theta^{\square}(a) \cap A) \cup (\theta^{\square}(a) \cap (B \setminus A)),$$
and $(\theta^{\square}(a) \cap (B \setminus A))$ is empty thanks to $\theta(A \setminus B) \cap (B \setminus A) = \emptyset$. Thus,
$$\theta^{\square}_{A\cup B}(z) = \theta^{\square}_{A}(a),$$
which is a (smooth) 0-PCM.
- When z belongs to $\text{Int}(A)$,
$$\begin{aligned}\theta^{\square}_{A\cup B}(z) &= (\theta^{\square}(z) \cap A) \cup (\theta^{\square}(z) \cap (B \setminus A))\\ &= (\theta^{\square}(z) \cap A) \cup (\theta^{\square}(z) \cap (B \setminus \{h\})),\end{aligned}$$
and $(\theta^{\square}(z) \cap (B \setminus \{h\}))$ is an empty set since $z \in \text{Int}(A) \subseteq (A \setminus B)$ and $\theta(A \setminus B) \cap (B \setminus A) = \emptyset$. Thus, $\theta^{\square}_{A\cup B}(z) = \theta^{\square}_{A}(z)$ which is a 0-surface.
- When z belongs to $A \cap B = \{h\}$, there exist $a' \in \text{Int}(A)$ and $b' \in \text{Int}(B)$ satisfying:
$$\theta^{\square}_{A\cup B}(z) = \theta^{\square}_{A}(h) \cup \theta^{\square}_{B}(h) = \{a', b'\},$$
and $a' \notin \theta(b')$ since $\theta(\text{Int}(A)) \cap \text{Int}(B) = \emptyset$. Thus, $\theta^{\square}_{A\cup B}(z)$ is a 0-surface.
- When $z \in \text{Int}(B)$, we obtain by symmetry that $\theta^{\square}_{A\cup B}(z)$ is a 0-surface.
- When $z = b$, we obtain by symmetry that $\theta^{\square}_{A\cup B}(z)$ is a (smooth) 0-PCM.

Finally, $\Delta(A \cup B) = \{a, b\}$, which is a 0-surface, thus $A \cup B$ is a (smooth) 1-PCM.

Heredity ($n \geq 2$): we assume that when A', B' are two smooth $(n-1)$-PCMs, with $A' \cap B' = \Delta A' \cap \Delta B'$ a smooth $(n-2)$-PCM, and the rank of $A' \cup B'$ is equal to $(n-1)$, $\theta(A' \setminus B') \cap (B' \setminus A') = \emptyset$, and $\theta(\text{Int}(A')) \cap \text{Int}(B') = \emptyset$, then $A' \cup B'$ is an $(n-1)$-PCM (induction hypothesis).

Let A, B be two smooth n-PCMs satisfying the conditions announced before, we are going to prove case-by-case that for any $z \in A \cup B$, $\theta^{\square}_{A\cup B}(z)$ is either a smooth $(n-1)$-PCM or an $(n-1)$-surface, with $\Delta(A \cup B) \neq \emptyset$. So, there are the different cases:

- When $z \in \Delta A \setminus B$,

  $$\theta^{\Box}_{A\cup B}(z) = (\theta^{\Box}(z) \cap A) \cup (\theta^{\Box}(z) \cap (B \setminus A)) = \theta^{\Box}_{A}(z),$$

  which is a smooth $(n-1)$-PCM. Additionally, we obtain that $\Delta(A \cup B) \neq \emptyset$.
- When $z \in \mathrm{Int}(A)$,

  $$\theta^{\Box}_{A\cup B}(z) = (\theta^{\Box}(z) \cap A) \cup (\theta^{\Box}(z) \cap (B \setminus A)) = \theta^{\Box}_{A}(z),$$

  which is an $(n-1)$-surface.
- When $z \in \Delta(A \cap B) = \Delta(\Delta A \cap \Delta B) \subseteq \Delta A \cap \Delta B$,

  $$\theta^{\Box}_{A\cup B}(z) = \theta^{\Box}_{A}(z) \cup \theta^{\Box}_{B}(z) = A' \cup B'$$

  when we denote by A' the term $\theta^{\Box}_{A}(z)$ and by B' the term $\theta^{\Box}_{B}(z)$. However, we know that A' and B' are smooth $(n-1)$-PCMs, that the rank of $A' \cup B'$ is $(n-1)$ (because if its rank is greater than $(n-1)$, the rank of $A \cup B$ would be $(n+1)$, which contradicts the hypotheses). We also know by Proposition 18 that:

  $$\begin{aligned} A' \cap B' &= \theta^{\Box}_{A\cap B}(z) \\ &= \theta^{\Box}_{\Delta A\cap \Delta B}(z) \\ &= \Delta\theta^{\Box}_{A}(z) \cap \Delta\theta^{\Box}_{B}(z) \\ &= \Delta A' \cap \Delta B', \end{aligned}$$

  and

  $$\begin{aligned} &\theta(A' \setminus B') \cap (B' \setminus A') \\ &\subseteq \theta(\theta^{\Box}(z) \cap (A \setminus B)) \cap (\theta^{\Box}(z) \cap (B \setminus A)) \\ &\subseteq \theta(A \setminus B) \cap (B \setminus A) \\ &= \emptyset, \end{aligned}$$

  and by Proposition 19:

  $$\theta(\mathrm{Int}(\theta^{\Box}_{A}(z)) \cap \mathrm{Int}(\theta^{\Box}_{B}(z)) \subseteq \theta(\mathrm{Int}A) \cap \mathrm{Int}B = \emptyset.$$

  Also, $A' \cap B' = \theta^{\Box}_{A\cap B}(z)$ with $A \cap B$ a smooth $(n-1)$-PCM. Since z belongs to $\Delta(A \cap B)$, $\theta^{\Box}_{A\cap B}(z)$ is a smooth $(n-2)$-PCM. Thus, the induction hypothesis applies, and $\theta^{\Box}_{A\cup B}(z)$ is a smooth $(n-1)$-PCM.
- When $z \in \mathrm{Int}(A \cap B) = \mathrm{Int}(\Delta A \cap \Delta B) \subseteq \Delta A \cap \Delta B$,

  $$\theta^{\Box}_{A\cup B}(z) = \theta^{\Box}_{A}(z) \cup \theta^{\Box}_{B}(z) = A' \cup B'$$

  when we denote by A' the term $\theta^{\Box}_{A}(z)$ and by B' the term $\theta^{\Box}_{B}(z)$. We are going to use Theorem 20 on A' and B':

  - A' and B' are smooth $(n-1)$-PCMs since $z \in \Delta A \cap \Delta B$,
  - $\mathrm{rk}(A' \cup B') = (n-1)$ (since $\mathrm{rk}(A' \cup B') \geq n$ implies $\mathrm{rk}(A \cup B) > n$ which is impossible),
  - $\Delta A' = \Delta B' = A' \cap B'$: Since:

    $$\theta^{\Box}_{\Delta A\cap \Delta B}(z) = \theta^{\Box}_{\Delta A}(z) \cap \theta^{\Box}_{\Delta B}(z),$$

    where:

    * $\theta^{\Box}_{\Delta A\cap \Delta B}(z)$ is contained in $\theta^{\Box}_{\Delta A}(z)$ which is an $(n-2)$-surface because ΔA is an $(n-1)$-surface,
    * $\theta^{\Box}_{\Delta A\cap \Delta B}(z)$ is contained in $\theta^{\Box}_{\Delta B}(z)$ which is an $(n-2)$-surface because ΔB is an $(n-1)$-surface,
    * $\theta^{\Box}_{\Delta A\cap \Delta B}(z)$ is an $(n-2)$-surface because z belongs to $\mathrm{Int}(\Delta A \cap \Delta B) = \mathrm{Int}(A \cap B)$ with $A \cap B$ a smooth $(n-1)$-PCM,

    which implies by Proposition 3 that:

    $$\theta^{\Box}_{\Delta A\cap \Delta B}(z) = \theta^{\Box}_{\Delta A}(z) = \theta^{\Box}_{\Delta B}(z),$$

    thus by Proposition 18:

    $$\Delta A' = \Delta\theta^{\Box}_{A}(z) = \Delta\theta^{\Box}_{B}(z) = \Delta B',$$

    and $\theta^{\Box}_{\Delta A\cap \Delta B}(z) = \theta^{\Box}_{A}(z) \cap \theta^{\Box}_{B}(z) = A' \cap B'$, we have then the equality:

    $$A' \cap B' = \Delta A' = \Delta B',$$

    or in other words:

    $$A' \cap B' = \Delta A' \cap \Delta B'.$$

  - $\theta(\mathrm{Int}(A')) \cap \mathrm{Int}(B') \subseteq \theta(\mathrm{Int}(A)) \cap \mathrm{Int}(B) = \emptyset$ by Proposition 19,

  Thus $A' \cup B' = \theta^{\Box}_{A\cup B}(z)$ is an $(n-1)$-surface.
- The case $z \in \mathrm{Int}B$ is obtained by symmetry and leads to $\theta^{\Box}_{A\cup B}(z)$ is an $(n-1)$-surface.
- The case $z \in \Delta B \setminus A$ is obtained by symmetry and leads to $\theta^{\Box}_{A\cup B}(z)$ is a smooth $(n-1)$-PCM.

Thus, $A \cup B$ is an n-PCM.

We obtain naturally:

$$\Delta(A \cup B) = (\Delta A \setminus B) \cup \Delta(A \cap B) \cup (\Delta B \setminus A).$$

The proof is done. $\square$

A.3 Cutting discrete surfaces

Theorem 22. *Let X be an n-surface with $n \geq 1$, and let N be some suborder of X which is an $(n-1)$-surface. We denote then by $\{C_i\}_{i\in\mathcal{I}}$ the connected components of $\mathcal{CC}(X \setminus N)$. We assume that $\#(\mathcal{I}) \geq 2$. Then, the components $|C_i \sqcup N|$ are smooth n-PCMs.*

Proof: Let us prove first that for any $i \in \mathcal{I}$, we have the relation:

$$\theta(C_i) \subseteq C_i \sqcup N.$$

If $\theta(C_i) \not\subseteq C_i \sqcup N$, there exists $h \in \theta(C_i)$ with $h \notin C_i \sqcup N$, then $h \in C_j$ with $j \in \mathcal{I}$ and $i \neq j$. Then C_i and C_j are neighbors, then equal (by definition). We obtain a contradiction, then $\theta(C_i) \subseteq C_i \sqcup N$ for any $i \in \mathcal{I}$.

Now let us prove by induction on $n \geq 1$ that when we decompose an n-surface X such that:

$$X = \bigcup_{i\in\mathcal{I}} C_i \sqcup N,$$

with:

$$\bigcap_{i\in\mathcal{I}} (C_i \sqcup N) = N,$$

with N an $(n-1)$-surface, each $C_i \sqcup N$ is a smooth n-PCM. We denote this property by $(\mathcal{P}_n)$.

Initialization $(n = 1)$: when X is a 1-surface and $N \subset X$ is a 0-surface, we decompose X into $\{C_1, C_2\} = \mathcal{CC}(X \setminus N)$ and they satisfy:

$$X = (C_1 \sqcup N) \cup (C_2 \sqcup N)$$

and:

$$(C_1 \sqcup N) \cap (C_2 \sqcup N) = N.$$

At this moment, we obtain that each term $(C_i \sqcup N)$ is a 1-PCM. Thus, $(\mathcal{P}_1)$ is true.

Heredity $(n \geq 2)$: we assume that $(\mathcal{P}_k)$ is true for any $k \in [\![1, n-1]\!]$, let us prove $(\mathcal{P}_n)$. We assume that X is an n-surface. Also, we set:

$$\{C_i\}_{i\in\mathcal{I}} = \mathcal{CC}(X \setminus N),$$

then we obtain:

$$\bigcup_{i\in\mathcal{I}} C_i \sqcup N = X,$$

and:

$$\bigcap_{i\in\mathcal{I}} C_i \sqcup N = N.$$

Let us prove that for any $i \in \mathcal{I}$ and for any $h \in C_i \sqcup N$, $|\theta^{\square}_{C_i\sqcup N}(h)|$ is either an $(n-1)$-surface or a smooth $(n-1)$-PCM. Let fix some $i \in \mathcal{I}$ and let h be an element of $C_i \sqcup N$, we have two possible cases:

- When $h \in C_i$, then $\theta^{\square}_X(h) \subseteq C_i \sqcup N$, then $\theta^{\square}_X(h) = \theta^{\square}_X(h) \cap (C_i \sqcup N)$, then $\left|\theta^{\square}_{C_i\sqcup N}(h)\right| = \left|\theta^{\square}_X(h)\right|$ is an $(n-1)$-surface since X is an n-surface.
- When $h \in N$, we remark that:

$$\begin{aligned}
\theta^{\square}_X(h) &= \theta^{\square}_X(h) \cap \bigcup_{i\in\mathcal{I}} (C_i \sqcup N) \\
&= \bigcup_{i\in\mathcal{I}} \theta^{\square}_X(h) \cap (C_i \sqcup N) \\
&= \bigcup_{i\in\mathcal{I}} \theta^{\square}_{C_i\sqcup N}(h),
\end{aligned}$$

with:

$$\begin{aligned}
\bigcap_{i\in\mathcal{I}} \theta^{\square}_{C_i\sqcup N}(h) &= \bigcap_{i\in\mathcal{I}} \theta^{\square}_X(h) \cap (C_i \sqcup N) \\
&= \theta^{\square}_X(h) \cap \bigcap_{i\in\mathcal{I}} (C_i \sqcup N) \\
&= \theta^{\square}_N(h).
\end{aligned}$$

In other words, the union of the terms $\left|\theta^{\square}_{C_i\sqcup N}(h)\right|$ is an $(n-1)$-surface, their intersection is $\left|\theta^{\square}_N(h)\right|$ and is an $(n-2)$-surface. Thus, by $(\mathcal{P}_{n-1})$, each term $\left|\theta^{\square}_{C_i\sqcup N}(h)\right|$ is a smooth $(n-1)$-PCM.

Adding the fact that $C_i \sqcup N$ is connected and that $\Delta(C_i \sqcup N) = N$ which is an $(n-1)$-surface by hypothesis, we conclude that each $|C_i \sqcup N|$ is a smooth n-PCM. Consequently, $(\mathcal{P}_n)$ is true.

This concludes the proof. $\square$

A.4 Cardinality theorem

Theorem 23. *Let $n \geq 1$ be some integer. Let X be a discrete n-surface and let $N \subset X$ be a* *discrete $(n-1)$-surface. Let us denote $\{C_i\}_{i \in \mathcal{I}} = \mathcal{CC}(X \setminus N)$. Then, the cardinality of $\mathcal{I}$ is equal to or lower than two. In other words, the "cut" of a discrete surface using a discrete surface of lower rank leads to at most two smooth n-PCM's.*

Proof: Let us assume that the cardinality of $\mathcal{I}$ is equal to or greater than 3. We can then write $\mathrm{Card}\,(\mathcal{I}) \geq 3$. Using Theorem 22, we know that each $C_i \sqcup N$ is a smooth n-PCM. Now, let us denote $A = C_1 \sqcup N$ and $B = C_2 \sqcup N$. We have the following properties:

- $\mathrm{rk}(A \cup B) = n$ since $n = \mathrm{rk}(A) = \mathrm{rk}(B) \leq \mathrm{rk}(A \cup B) \leq \mathrm{rk}(X) = n$ (by increasingness of the operator rk),
- $\Delta A = \Delta B = N = A \cap B$,
- $\theta(\mathrm{Int}(A)) \cap \mathrm{Int}(B) = \theta(C_1) \cap C_2 = \emptyset$,

from which we can deduce by Theorem 20 that $A \cup B$ is an n-surface. However, $A \cup B \subsetneq X$ (since $C_3 \neq \emptyset$ is contained in X but does not intersect $A \cup B$), which means that we have two nested discrete n-surfaces which are different, it is a contradiction. Thus, the cardinality of $\mathcal{I}$ belongs to $\{1, 2\}$. The proof is done. $\square$

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