Posets with Interfaces as a Model for Concurrency

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Abstract

We introduce posets with interfaces (iposets) and generalise their standard serial composition to a new gluing composition. In the partial order semantics of concurrency, interfaces and gluing allow modelling events that extend in time and across components. Alternatively, taking a decompositional view, interfaces allow cutting through events, while serial composition may only cut through edges of a poset. We show that iposets under gluing composition form a category, which generalises the monoid of posets under serial composition up to isomorphism. They form a 2-category when a subsumption order and a lax tensor in the form of a non-commutative parallel composition are added, which generalises the interchange monoids used for modelling series-parallel posets. We also study the gluing-parallel hierarchy of iposets, which generalises the standard series-parallel one. The class of gluing-parallel iposets contains that of series-parallel posets and the class of interval orders, which are well studied in concurrency theory, too. We also show that it is strictly contained in the class of all iposets by identifying several forbidden substructures.

Keywords: Poset; interval order; series-parallel poset; concurrency theory; iposet; gluing-parallel iposet

1. Introduction

Our general motivation for studying posets with interfaces (iposets) comes from concurrent Kleene algebra [18]. There, such structures have been proposed as semantics for concurrent programs, because the compositionality inherent to standard partial order models limits their applicability (see [27] for a survey). Our particular conceptual choices of interfaces and operations on iposets are motivated by languages of higher-dimensional automata [10]. Our gluing

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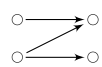
composition is meant to match the gluing of such automata along faces, our non-commutative parallel composition to correspond to their tensor product. Higher-dimensional automata generalise many other models of concurrency [29], which makes the study of their languages relevant to concurrent Kleene algebra.

Nevertheless, this article is not about Kleene algebras or higher-dimensional automata per se. Instead we study the basic order-theoretic and algebraic properties of iposets, explore their categorical structure and investigate their alternation hierarchy under parallel and gluing composition.

Our starting point are partial order semantics of concurrency [32], where points of finite posets model events of concurrent systems and partial order relations model temporal precedences or causal dependencies between events, as well as events that occur in parallel or independently. Events are often labelled with the actions of a concurrent system. One then tends to forget individual events and model only structural properties of labelled posets up to isomorphism. Isomorphism classes of labelled posets are known as *partial words* or *pomsets* [14,15,25,34]. Pomsets are usually equipped with two compositions. Intuitively, a parallel composition lays out one pomset above another. A serial one lays out one pomset to the left of another and extends their orders so that each event in the left pomset precedes every event in the right one. The series-parallel pomsets are then generated by the empty pomset and all one-event pomsets, and closed under serial and parallel compositions. It is a strict subclass of the class of all pomsets, because it excludes precisely those pomsets that contain an induced subposet of the shape

N =  .

Finally, a subsumption order compares pomsets with the same set of events in terms of their precedence relations [14,15]. For so-called *concurrent monoids*, that is, ordered double monoids with shared units in which the serial composition $\cdot$ and the commutative parallel composition $\times$ interact via the lax interchange law

$$(x_1 \times x_2) \cdot (y_1 \times y_2) \leq (x_1 \cdot y_1) \times (x_2 \cdot y_2),$$

the series-parallel pomsets over the set A of labels and with subsumption order $\leq$ are then freely generated by A in this class of algebras [2,14].

Here we adapt these notions and results to iposets. As we are mainly interested in the order structure, we ignore labels and tacitly work with isomorphism classes of finite posets.

One motivation for our use of interfaces comes from modeling events with duration or structure (by contrast to interleaving models of concurrency where they are instantaneous and indivisible). In the standard partial order semantics of concurrency we may interpret the order on events as temporal precedence. A concurrent system can then be decomposed into two parallel parts if and only if these are temporally disconnected. Similarly, it can be serially decomposed if and only if all events in the first component precede all events in the second. In iposets, more generally, we allow serial decompositions to *cut through events* as well, so that the extend from the first component into the second. Such events then form the target interface of the first component and the source interface of the second. See [23, 26] for further discussion and physical interpretations.

To match this decompositional view with algebraic operations, the source interface of an iposet is formed by some of its minimal events (unless it is empty) and the target interface by some of its maximal ones. Our gluing composition acts like standard serial composition outside of interfaces, but gluing interface events together. The particular matching of interface events is determined by numbering them, and this and the resulting renumbering makes parallel composition non-commutative. The gluing composition has many units, whereas the parallel composition has a single one—the empty poset. Yet both operations are associative. The standard serial and parallel compositions of posets are recovered when iposets have empty interfaces.

Using these basic algebraic properties we show that iposets with gluing and parallel compositions and a suitable generalisation of the subsumption order form a strict 2-category with parallel composition as a lax tensor. This yields a lax interchange law between gluing and parallel composition. The notion of strict 2-categories with lax tensors is new, but closely related to standard 2-categorical structures. It generalises the concurrent monoids that capture the equational theory of series-parallel pomsets.

By analogy to series-parallel posets, we also define a hierarchy of gluing-parallel iposets and show that it does not collapse. We identify forbidden substructures in order to show that it is a proper subclass of the class of all iposets. We relate it with the hierarchy of series-parallel pomsets as well as with interval orders. All series-parallel posets are in the gluing-parallel hierarchy. The interval orders, which allow N-shaped posets, are captured at its first alternation level. Interval orders arise naturally in geometric realisation of higher-dimensional automata or in situations where events in concurrent systems extend in time [23, 26, 31], such as weak and transactional memory systems [17]. A precise geometric characterisation of gluing-parallel iposets in terms of forbidden substructures remains open. Further, the precise algebraic setting in which they would be freely generated, analogous to series-parallel posets being free concurrent monoids, is open. (The 2-category outlined above does not suffice: certain iposets satisfy a strong interchange law.)

This article is based on a previous conference paper [9]. The description of iposets in terms of strict 2-categories in Definition 6 and Theorem 5 is one of them. A sufficient condition for the existence of gluing decompositions in Lemma 11 is another one. We also make the hierarchies in Section 9 more precise and add several new forbidden substructures in Proposition 32. An erroneous claim about freeness in a particular algebraic class [9, Thm. 19] is refuted in Example 4.

2. Posets

We assume basic knowledge of order theory, see [7] for details. We restrict our attention to finite posets with strict orderings, as is usual in the partial order semantics of concurrency. Throughout this paper, a poset $(P, <)$ is thus a finite set P equipped with an irreflexive transitive binary relation $<$ (asymmetry of $<$ follows). In Hasse diagrams, we put greater elements to the right of smaller ones. In the opposite poset $P^{\mathrm{op}} = (P, >)$ of a poset $(P, <)$, the order is reversed.

We write $[n] = \{1, \dots, n\}$, for $n \geq 1$, both for the set with elements 1 to n and for the discrete poset on these points where all points are incomparable. Additionally, $[0] = \emptyset$. We write $[\vec{n}] = \{1 < \cdots < n\}$ for the linear order on $[n]$. In particular, $[\vec{1}] = [1]$.

A function $f : P \to Q$ between posets $(P, <_P)$ and $(Q, <_Q)$ is order-preserving if $f(x) <_Q f(y)$ whenever $x <_P y$, and order-reflecting if $x <_P y$ whenever $f(x) <_Q f(y)$. A poset monomorphism is an order-preserving and order-reflecting injection. A poset P is an induced subposet of Q if there exists a monomorphism $P \hookrightarrow Q$. Poset isomorphisms are bijective monomorphisms; we write $P \cong Q$ if P and Q are isomorphic.

We usually consider posets, and operations on them, up to isomorphism. Intuitively this means that we are not interested in the identity of the events in a poset $(P, <)$ but rather in their order structure given by precedence relation $<$. The set of isomorphism classes of posets is denoted by Pos.

Remark 1. By contrast to $\leq$-embeddings, $<$-preserving and reflecting functions need not be injective: the only $<$-preserving function from the V-shape three-point poset to $[\vec{2}]$ is $<$-reflecting (and surjective), but has no inverse. Every $<$-preserving function is nevertheless injective on comparable points.

In concurrency theory, posets are often equipped with a serial and a parallel composition [34]. Both are based on the disjoint union (coproduct) of sets, defined as $X \sqcup Y = \{(x, 1) \mid x \in X\} \cup \{(y, 2) \mid y \in Y\}$.

Definition 1. Let $(P_1, <_1)$ and $(P_2, <_2)$ be posets.

1. Their *parallel composition* $P_1 \otimes P_2$ is the coproduct with $P_1 \sqcup P_2$ as carrier set and order defined as

$$(p, i) < (q, j) \;\Leftrightarrow\; i = j \wedge p <_i q, \qquad i, j \in \{1, 2\}.$$

2. Their *serial composition* $P_1 * P_2$ is the ordinal sum, which again has the disjoint union as carrier set, but order defined as

$$(p, i) < (q, j) \;\Leftrightarrow\; (i = j \wedge p <_i q) \vee i < j, \qquad i, j \in \{1, 2\}.$$

Intuitively, $P_1 \otimes P_2$ puts the Hasse diagram of P_1 above that of P_2, whereas $P_1 * P_2$ puts the Hasse diagram of P_1 to the left of that of P_2 and adds arrows from each element of P_1 to each element of P_2. It is clear that both operations are well defined on isomorphism classe. They are associative and have the empty poset as their unit, up to isomorphism. Parallel composition is commutative while serial composition is not. Isomorphism classes of posets thus form a monoid with respect to serial composition and a commutative monoid with respect to concurrent composition. These monoids share their unit (the empty pomset).

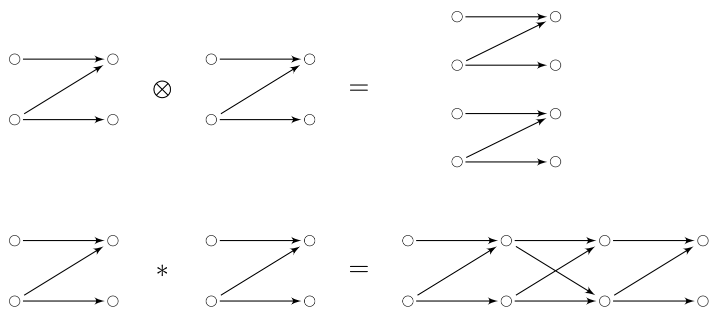


Figure 1: Parallel and serial compositions of posets N.

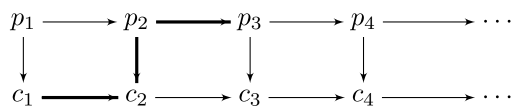


Figure 2: Producer-consumer pomset with an induced subposet N indicated by bold arrows.

Example 1. For any $n, m \geq 0$, it holds that $[n] \otimes [m] \cong [n+m]$ and $[\overrightarrow{n}] * [\overrightarrow{m}] \cong [\overrightarrow{n+m}]$; thus $[n] \cong [1] \otimes \cdots \otimes [1]$ and $[\overrightarrow{n}] \cong [1] * \cdots * [1]$ (n times each). Figure 1 shows further examples.

A poset is *series-parallel* (an *sp-poset*) if it is either empty or generated from the singleton poset by finitely many serial and parallel compositions. As mentioned in the introduction, sp-posets are precisely those posets that do not contain N as an induced subposet.

This makes sp-posets unsuitable for many applications: even simple producer-consumer systems generate N's [22] and their structure cannot be captured by sp-posets, see Figure 2.

Interval orders [12,33] form another class of posets relevant to concurrent and distributed computing [19]. Intuitively, they are isomorphic to sets of intervals on the real line that are ordered whenever they do not overlap. Interval orders can therefore capture events that extend in time.

Definition 2. An *interval order* is a poset $(P, <)$ such that $w < y$ and $x < z$ imply $w < z$ or $x < y$, for all $w, x, y, z \in P$.

(Transitivity of $<$ follows from asymmetry and the property above.) Geometrically, there is once again a single forbidden substructure: interval orders are precisely those posets that do not contain an induced subposet of the form

$$\mathsf{2+2} = [\overrightarrow{2}] \otimes [\overrightarrow{2}] =$$

.

The intuition outlined above is captured by Fishburn's representation theorem [12, 13]. A poset P is an interval order if and only if it has an *interval* *representation*: a map $I : P \to 2^{\mathbb{R}}$ assigning to each $x \in P$ a closed real interval $I(x) = [b(x), e(x)]$, with $b(x) \leq e(x)$, such that $y <_P z$ if and only if $e(y) <_{\mathbb{R}} b(z)$, for all $y, z \in P$.

Each interval order admits an interval representation with a minimal number of endpoints. Such minimal representations are in bijective correspondence with the *closed interval traces* of concurrency theory [20], as shown in [9]. Closed interval sequences are finite sequences of $b(x)$ and $e(x)$ with x ranging over some finite set, where each $b(x)$ and $e(x)$ occurs exactly once and each $e(x)$ after the corresponding $b(x)$. Closed interval traces are equivalence classes of such sequences modulo the relations $b(x)b(y) \approx b(y)b(x)$ and $e(x)e(y) \approx e(y)e(x)$. They are order-theoretic analogues of *ST-traces* of Petri nets [28, 32].

We show in Section 9 that interval orders appear at the first level of the alternation hierarchy of the gluing-parallel posets with interfaces introduced in the following sections.

3. Posets with Interfaces

We now define posets with interfaces, their gluing and parallel composition and the units of these operations. We show that posets with interfaces, up to isomorphism, form a category with respect to gluing composition and that the presence of interfaces makes our parallel composition non-commutative.

Definition 3. A *poset with interfaces* (*iposet*) is a poset $(P, <)$ equipped with two injective morphisms

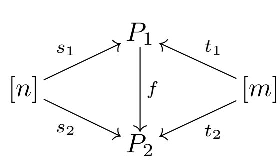
$$[n] \xrightarrow{s} P \xleftarrow{t} [m], \qquad n, m \geq 0,$$

such that the elements in the image $s([n])$ are minimal and those in $t([m])$ maximal in P.

The injections $[n] \overset{s}{\to} P \overset{t}{\leftarrow} [m]$ represent the *source* and the *target interface* of P, respectively. We write S_P instead of $s_P([n])$ and T_P instead of $t_P([m])$ for sets of interface elements and drop indices if convenient. We also write $(s, P, t) : n \to m$ instead of $[n] \xrightarrow{s} P \xleftarrow{t} [m]$; even $P : n \to m$ when the interfaces are clear. The *opposite* of an iposet $(s, P, t) : n \to m$ is the iposet $(t, P^{\text{op}}, s) : m \to n$, also denoted by P^{op}, where the order has been reversed and the source and target interfaces have been swapped.

Figure 3 shows examples of iposets. Interface elements are represented as half-circles to indicate the incomplete nature of the corresponding events.

Definition 4. A *subsumption* of iposets $(s_1, P_1, t_1), (s_2, P_2, t_2) : n \to m$ is an order-reflecting bijection $f : P_1 \to P_2$ that preserves interfaces: $f(x) <_2 f(y)$ implies $x <_1 y$ for all $x, y \in P_1$, $f \circ s_1 = s_2$, and $f \circ t_1 = t_2$.



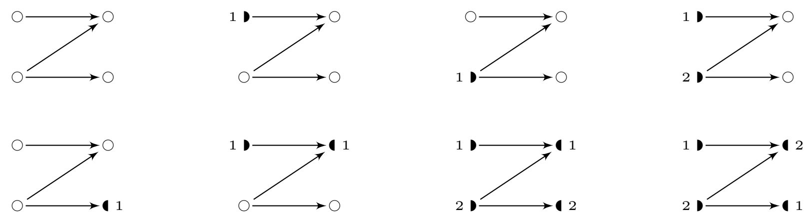


Figure 3: Eight of 25 different iposets based on poset N.

An iposet *isomorphism* is an order-preserving subsumption.

We write $P_1 \preceq P_2$ if there exists a subsumption $P_1 \to P_2$ and $P_1 \cong P_2$ if there exists an iposet isomorphism $P_1 \to P_2$. The relation $\preceq$ is a preorder and $\cong\, = \,\preceq \cap \succeq$ (proving this requires some thought [8]).

Remark 2. The subsumption relation $\preceq$ for posets is well studied [14, 15]. In the literature, $P_1 \preceq P_2$ is usually defined by existence of an order-preserving bijection $P_2 \to P_1$. This is equivalent to our definition.

Like for posets above, we usually consider iposets up to isomorphism. The subsumption relation $\preceq$ is a partial order on isomorphism classes. Intuitively, $P_1 \preceq P_2$ holds if P_1 and P_2 have the same points and interfaces and P_1 is at least as ordered as P_2.

Next we introduce a partial sequential gluing composition on iposets, defined whenever the interfaces on ends agree. We also adapt the standard parallel composition of posets to iposets. The latter requires the isomorphisms $\phi_{n,m} : [n+m] \to [n] \otimes [m]$ given by

$$\phi_{n,m}(i) = \begin{cases} (i,1) & \text{if } i \leq n, \\ (i-n,2) & \text{if } i > n; \end{cases}$$

we also tacitly use parallel composition of poset morphisms, which is defined in the obvious way.

Definition 5. Let $(s_1, P_1, t_1) : n_1 \to m_1$ and $(s_2, P_2, t_2) : n_2 \to m_2$ be iposets.

1. Their *parallel composition* is the iposet $P_1 \otimes P_2 = (s, P_1 \otimes P_2, t) : n_1 + n_2 \to m_1 + m_2$ with $s = (s_1 \otimes s_2) \circ \phi_{n_1,n_2}$ and $t = (t_1 \otimes t_2) \circ \phi_{m_1,m_2}$.

2. For $m_1 = n_2$, their *gluing composition* is the iposet $P_1 \triangleright P_2 = (s_1, P, t_2) : n_1 \to m_2$, where the carrier set is the quotient

$$P = (P_1 \sqcup P_2)/\{(t_1(k), 1) = (s_2(k), 2)\}_{k \in [m_1]}$$

and the order is defined, for $i, j \in \{1, 2\}$, as

$$(p,i) < (q,j) \Leftrightarrow (i = j \wedge p <_i q) \vee (i < j \wedge p \notin T_{P_1} \wedge q \notin S_{P_2}).$$

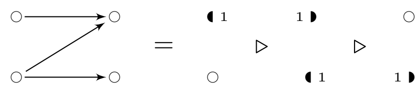


Figure 4: Decomposition of poset N.

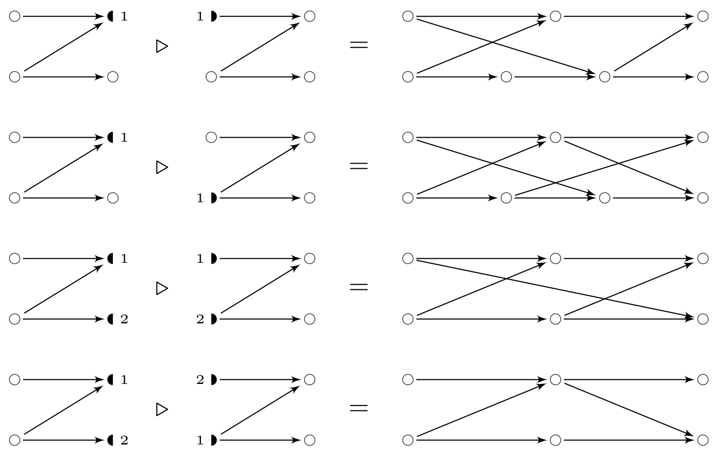


Figure 5: Gluings of different Ns with interfaces.

Intuitively, parallel composition of iposets puts the Hasse diagrams of the underlying posets on top of each other while renumbering interfaces. Gluing composition $P_1 \triangleright P_2$ puts the Hasse diagram of P_1 to the left of that of P_2, whenever the target interface of P_1 and the source interface of P_2 match. It then glues corresponding interface points and adds arrows from all points in P_1 that are not in its target interface to all points in P_2 that are not in its source interface. As explained in the introduction, it thus glues events of P_1 that do not finish in P_1 with those events of P_2 that do not start in P_2.

It is clear that $\otimes$ and $\triangleright$ are well defined on isomorphism classes: they are associative up to isomorphism. Figures 4 and 5 show examples of gluing (de)compositions of iposets, including the N. The half-circles in source and target interfaces are glued to full circles in these diagrams.

Proposition 1. *Iposets form a category* iPos *with natural numbers as objects, isomorphism classes of iposets* $(s, P, t) : n \to m$ *as morphisms, the* identity iposets $\mathsf{id}_n = (\mathsf{id}_{[n]}, [n], \mathsf{id}_{[n]}) : n \to n$ *as identity morphisms,* $\triangleright$ *as composition and dom, cod* $: \mathsf{iPos} \to \mathbb{N}$ *sending* $P : n \to m$ *to* n *and* m*, respectively, as domain and codomain maps.*

Proof. Checking the standard unit, domain and codomain axioms for categories is trivial; associativity of gluing composition (up to isomorphism) follows directly from unfolding Definition 5(2). □

Posets may be regarded as iposets with empty interfaces. Thus, as sets,

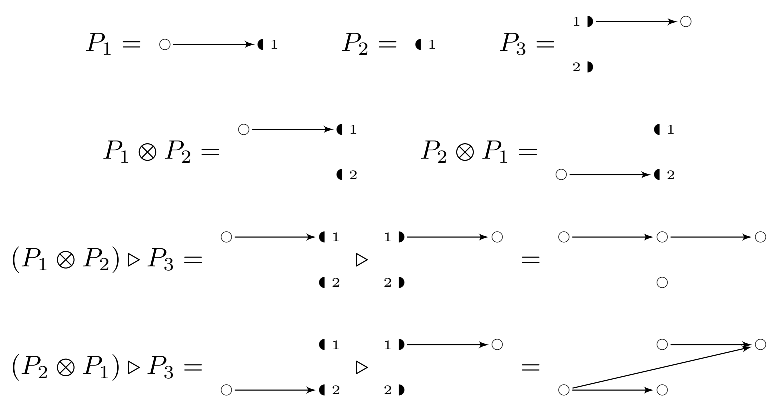
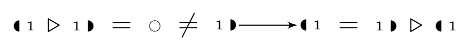


Figure 6: Non-isomorphic gluings of symmetric parallel compositions in Example 2.

Pos $\cong$ iPos$(0,0)$, where iPos$(0,0)$ denotes the iPos morphisms at object 0. We write gluing composition in diagrammatical order.

serial poset composition for posets, $\triangleright$ is not commutative:



Parallel composition has the empty iposet as its unit. As the next example shows, it need not commute.

Example 2. Figure 6 shows iposets P_1 and P_2 and their parallel products $P_1 \otimes P_2$ and $P_2 \otimes P_1$. The latter are non-isomorphic as iposets because of the different labelling of the interfaces. Yet their underlying posets are isomorphic. However, when they are glued with iposet P_3 in the figure, the resulting posets $(P_1 \otimes P_2) \triangleright P_3$ and $(P_2 \otimes P_1) \triangleright P_3$ are non-isomorphic *as posets*. Non-commutativity of parallel composition is therefore not a technical artefact of our definitions but inherent to the formalism.

Interfaces can be renumbered using special iposets called *symmetries*, as explained in Section 6. Lemma 9 in that section shows that parallel composition of iposets is commutative up to such symmetries; but following Example 2 above, enforcing such commutativity would make gluing composition lose its congruence property, which is undesirable. Further, Lemma 13 below shows that gluing-parallel iposets, the class of iposets we are mainly interested in, satisfy an interface consistency property that rules out all non-trivial symmetries. Proposition 15 and Lemma 17 below describe precisely those parallel compositions that commute.

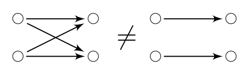
4. Interchange

This and the next sections study the interaction of gluing and parallel composition. For posets, serial and parallel composition interact through the lax interchange law

$$(P \otimes P') \triangleright (Q \otimes Q') \preceq (P \triangleright Q) \otimes (P' \triangleright Q').$$

It equips the set of isomorphism classes of posets with a concurrent monoid structure [18]. However, iposets with gluing composition form a category rather than a monoid; the interaction with parallel composition thus requires a (strict) 2-category. Readers unfamiliar with 2-categories can skip the following Section 5 which is not relevant to the rest of the paper.

A natural question is whether $\otimes$ yields a monoidal structure on iPos. Yet the answer is negative: gluing and parallel composition need not satisfy the interchange law a tensor$\otimes$ would require:

$(\circ\ \otimes\ \circ) \triangleright (\circ\ \otimes\ \circ) =$  $= (\circ\ \triangleright\ \circ) \otimes (\circ\ \triangleright\ \circ).$

Remark 3. This differs from gluing compositions where interfaces of iposets are defined by *all* minimal and maximal elements [34]. It also differs from previous serial compositions with interfaces [11, 24], where interfaces disappear and no additional order is introduced. The first case gives rise to strict monoidal categories with a partially defined tensor, the other to plain strict monoidal categories and, more specifically, PROPs.

The following proposition introduces a lax interchange law for iposets.

Proposition 2 (Lax interchange). *Let P, P', Q and Q' be iposets such that cod(P) = dom(Q) and cod(P') = dom(Q'). Then cod$(P \otimes P')$ = dom$(Q \otimes Q')$ and*

$$(P \otimes P') \triangleright (Q \otimes Q') \preceq (P \triangleright Q) \otimes (P' \triangleright Q'). \tag{1}$$

Proof. The first claim holds because

$$\mathit{cod}(P \otimes P') = \mathit{cod}(P) \otimes \mathit{cod}(P') = \mathit{dom}(Q) \otimes \mathit{dom}(Q') = \mathit{dom}(Q \otimes Q').$$

Set $L = (P \otimes P') \triangleright (Q \otimes Q')$ and $R = (P \triangleright Q) \otimes (P' \triangleright Q')$. For the underlying sets, the definition of $\otimes$ implies that

$$\begin{aligned} R &= (P \sqcup Q)_{/t_P(i)=s_Q(i)} \sqcup (P' \sqcup Q')_{/t_{P'}(j)=s_{Q'}(j)} \\ &= (P \sqcup Q \sqcup P' \sqcup Q')_{/t_P(i)=s_Q(i), t_{P'}(j)=s_{Q'}(j)} \\ &= (P \sqcup P' \sqcup Q \sqcup Q')_{/t_{P\otimes P'}(k)=s_{Q\otimes Q'}(k)} = L. \end{aligned}$$

Both posets thus have the same carrier set, and we may choose $f : L \to R$ to be the identity. It remains to show that f reflects the order: each arrow in R must be in L.

Suppose $x <_R y$, that is, $x <_{P\triangleright Q} y$ or $x <_{P'\triangleright Q'} y$. In the first case, if $x <_P y$ or $x <_Q y$, then $x <_{P\otimes P'} y$ or $x <_{Q\otimes Q'} y$ and therefore $x <_L y$; and if $x \in P \setminus T_P$ and $y \in Q \setminus S_Q$, then $x \in P \sqcup P' \setminus T_{P\otimes P'}$ and $y \in Q \sqcup Q' \setminus S_{Q\otimes Q'}$ and therefore $x <_L y$, too. The second case is symmetric, and $x <_L y$ holds. $\square$

5. The 2-Category of Iposets

We now introduce the 2-categorical generalisation of concurrent monoids which properly characterises our algebraic setting. Readers unfamiliar with 2-categories can skip this section.

First, recall that the subsumption morphism used in the proof of Proposition 2 is the identity on underlying sets; in anticipation of the 2-category structure introduced below we used 2-cell notation for subsumptions and isomorphisms of iposets, $f : P_1 \Rightarrow P_2$ instead of $f : P_1 \to P_2$, in this section only.

Corollary 3. *In the setting of Proposition 2, we have the subsumption*

$$\mathsf{id} : (P \otimes P') \rhd (Q \otimes Q') \Rightarrow (P \rhd Q) \otimes (P' \rhd Q').$$

Next we extend gluing composition to subsumptions.

Lemma 4. *Let $f : P \Rightarrow P'$ and $g : Q \Rightarrow Q'$ be subsumptions such that cod$(P) =$ dom(Q) and cod$(P') =$ dom(Q'). Then $h = f \rhd g : P \rhd Q \Rightarrow P' \rhd Q'$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in P, \\ g(x) & \text{if } x \in Q \end{cases}$$

is well defined and a subsumption.

Proof. Well-definedness of h follows from the fact that subsumptions preserve interfaces, and h is trivially a subsumption. □

We define a notion of lax tensor for (strict) 2-categories as a specialisation of the notion of lax functor between bicategories, see for example [21, Sect. 4.1]. Recall that 2-categories are formed by objects (0-cells), morphisms (1-cells) and 2-cells. Their axioms generalise those of categories. We write "$;$", in diagrammatical order, for composition in a general 2-category and "$\Rightarrow$" for 2-cells.

Definition 6. A *lax tensor* on a strict 2-category $\mathcal{C}$ is an operation $\otimes : \mathcal{C}_{0,1} \times \mathcal{C}_{0,1} \to \mathcal{C}_{0,1}$ on the $(0,1)$-restriction of $\mathcal{C}$, together with an object $\iota \in \mathcal{C}_0$ and 2-cells

$$I_{f_1,g_1,f_2,g_2} : (f_1 \otimes f_2);(g_1 \otimes g_2) \Rightarrow (f_1;g_1) \otimes (f_2;g_2)$$

for all morphisms $f_1 : a_1 \to b_1$, $g_1 : b_1 \to c_1$, $f_2 : a_2 \to b_2$, $g_2 : b_2 \to c_2$, such that

1. $f \otimes g : a \otimes c \to b \otimes d$ for all morphisms $f : a \to b$ and $g : c \to d$;
2. $\mathsf{id}_a \otimes \mathsf{id}_b = \mathsf{id}_{a \otimes b}$ for all objects a and b;
3. $\iota \otimes a = a \otimes \iota = a$ and $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ for all objects a, b, c;
4. $\mathsf{id}_\iota \otimes f = f \otimes \mathsf{id}_\iota = f$ and $f \otimes (g \otimes h) = (f \otimes g) \otimes h$ for all morphisms f, g, h;

5. as a mapping from tuples (f_1, g_1, f_2, g_2) of morphisms to 2-cells, I is natural, *i.e.*, the diagram

$$\begin{array}{ccc} \mathcal{C}(a_1,b_1)\times\mathcal{C}(a_2,b_2)\times\mathcal{C}(b_1,c_1)\times\mathcal{C}(b_2,c_2) & \xrightarrow{\;;\,\times\,;\;} & \mathcal{C}(a_1,c_1)\times\mathcal{C}(a_2,c_2) \\ \Big\downarrow{\scriptstyle \otimes\times\otimes} & \overset{I}{\nearrow\!\!\!\!\nearrow} & \Big\downarrow{\scriptstyle\otimes} \\ \mathcal{C}(a_1\otimes a_2,b_1\otimes b_2)\times\mathcal{C}(b_1\otimes b_2,c_1\otimes c_2) & \xrightarrow[\;;\;]{} & \mathcal{C}(a_1\otimes a_2,c_1\otimes c_2)\end{array}$$

commutes;

6. and I satisfies lax associativity:

$$\begin{array}{ccc} (f_1\otimes f_2);(g_1\otimes g_2);(h_1\otimes h_2) & \xRightarrow{\mathsf{id}_{f_1\otimes f_2};\,I_{g_1,h_1,g_2,h_2}} & (f_1\otimes f_2);((g_1;h_1)\otimes(g_2;h_2)) \\ \Big\Downarrow{\scriptstyle I_{f_1,g_1,f_2,g_2};\,\mathsf{id}_{h_1\otimes h_2}} & & \Big\Downarrow{\scriptstyle I_{f_1,g_1;h_1,f_2,g_2;h_2}} \\ ((f_1;g_1)\otimes(f_2;g_2));(h_1\otimes h_2) & \xRightarrow[I_{f_1;g_1,h_1,f_2;g_2,h_2}]{} & (f_1;g_1;h_1)\otimes(f_2;g_2;h_2)\end{array}$$

A lax tensor on $\mathcal{C}$ is thus precisely a lax and strictly unital functor $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$. The "interchanger" 2-cells I_{f_1,g_1,f_2,g_2} replace the identities in the strict interchange law, but are not required to be invertible.

We also need the following generalisation of acyclic or loop-free categories [3,16].

Definition 7. A 2-category $\mathcal{C}$ is 2-*acyclic* if it holds for all $f, g \in \mathcal{C}_1$ that $\mathcal{C}_2(f,g) \neq \emptyset$ and $\mathcal{C}_2(g,f) \neq \emptyset$ imply $f = g$ and $\mathcal{C}_2(f,g) = \{\mathsf{id}_f\}$.

Theorem 5. *With the data given below, iposets form the* 2*-acyclic strict* 2*-category* iPos *with* $\otimes$ *as a lax tensor:*

- *objects: natural numbers;*
- *morphisms* $n \to m$*: iposets* $(s, P, t) : n \to m$*;*
- *composition:* $\rhd : (n \to m) \times (m \to k) \to (n \to k)$*;*
- *identities:* $\mathsf{id}_n : n \to n$*;*
- 2*-cells* $(n \to m) \Rightarrow (n \to m)$*: subsumptions;*
- *vertical* 2*-composition: function composition of subsumptions;*
- *vertical* 2*-identities:* $\mathsf{id} : (n \to m) \Rightarrow (n \to m)$*;*

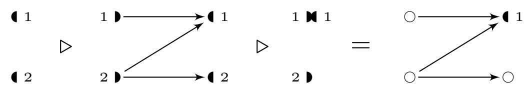


Figure 7: Switching off interfaces by composition with a starter and a terminator.

- *horizontal* 2*-composition: gluing composition of subsumptions;*
- *horizontal* 2*-identities:* $\mathsf{id} : \mathsf{id}_n \Rightarrow \mathsf{id}_n$.

Proof. The only properties of iPos that remain to be shown are associativity of horizontal 2-composition and the fact that all $\mathsf{id} : \mathsf{id}_n \Rightarrow \mathsf{id}_n$ are horizontal 2-identities. Both are trivial. Also 2-acyclicity is clear given that $P \preceq Q$ and $Q \preceq P$ imply $P \cong Q$.

To show that $\otimes$ is a lax tensor, Corollary 3 implies that the interchanger 2-cells $I_{P,Q,P',Q'} : (P \otimes P') \triangleright (Q \otimes Q') \Rightarrow (P \triangleright Q) \otimes (P' \triangleright Q')$ are the identities on the underlying sets. Naturality and lax associativity are then immediate. □

The 2-category iPos generalises the concurrent monoids mentioned in the introduction. The following proposition makes this relationship precise.

Proposition 6. *Let $\mathcal{C}$ be a 2-acyclic strict 2-category with lax tensor such that $\mathcal{C}_0$ contains precisely one object. With order defined by existence of 2-cells, $\mathcal{C}_1$ forms a concurrent monoid.*

Proof. The $\otimes$-unit $\iota \in \mathcal{C}_0$ is the unique object, and ; and $\otimes$ are operations on $\mathcal{C}_1 = \mathcal{C}_1(\iota, \iota)$. As $\mathcal{C}_{0,1}$ is a category, ; is associative and has ι as unit; by item 3 of Definition 6, the same is true for $\otimes$. Let $\leq$ be the preorder on $\mathcal{C}_1$ given by $f \leq g$ if $\mathcal{C}_2(f, g) \neq \emptyset$, then $\leq$ is a partial order as $\mathcal{C}_2$ is 2-acyclic, and $(f \otimes g); (h \otimes k) \leq (f; h) \otimes (g; k)$ for all $f, g, h, k \in \mathcal{C}_1$ because of item 6 of Definition 6. □

In the setting of Theorem 5, iposet isomorphisms are 2-cells. They should rightly be called 2-isomorphisms; but we will keep the simpler terminology.

6. Starters, Terminators and Symmetries

This section and the next prepare our treatment of iposets generated from singletons in Section 8. Here we discuss properties of discrete iposets, which are important for defining non-trivial (de)compositions of iposets and for understanding symmetries.

An iposet is *discrete* if its order is empty. A discrete iposet $(s, P, t) : n \to m$ is a *starter* if $t : [m] \to P$ is bijective. By opposition, it is a *terminator* if $s : [n] \to P$ is bijective.

Starters and terminators are useful for starting and terminating individual events in gluing compositions: in $Q = S \triangleright P \triangleright T$ with starter S and terminator T, the poset of Q equals that of P, but parts of the interfaces of P may have

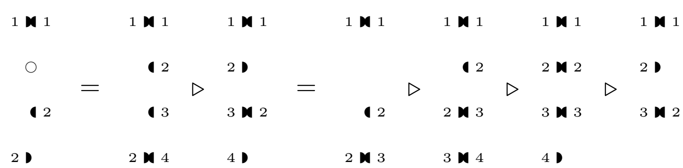


Figure 8: Decomposition of discrete iposet into starters and terminators.

been switched off by S and T. As an example, Figure 7 shows how a starter is used to start the minimal events of an iposet N in which they are already running, and a terminator is used to terminate one unfinished event of the N.

Every discrete iposet can obviously be obtained by gluing a starter with a terminator, see also Figure 8.

Lemma 7. *Let $P : n \to m$ and $Q : m \to \ell$ be iposets. Then $P \triangleright Q \cong Q$ as posets if and only if P is a starter, and $P \triangleright Q \cong P$ as posets if and only if Q is a terminator.*

Proof. We only show the first claim; the second one follows by opposition. If P is a starter, then obviously $P \triangleright Q \cong Q$ as posets. Conversely, if $P \triangleright Q \cong Q$ as posets, then $(P \sqcup Q)_{/t_P(i)=s_Q(i)} = Q$. Thus t_P must be surjective and therefore bijective. □

In light of Lemma 7 we may now define when a gluing (de)composition is non-trivial. This is easy for *posets* where $P * Q$ is trivial precisely when P or Q is empty; but the presence of interfaces complicates matters.

Definition 8. A gluing product $P \triangleright Q$ is *trivial* if P is a starter or Q a terminator.

Every identity is obviously a starter and a terminator, but not every iposet that is a starter and a terminator is an identity. A *symmetry* is a discrete iposet $(s, P, t) : n \to n$ that is both a starter and a terminator. Thus s and t are both bijections. All points of P are in the starting and terminating interfaces, but $t^{-1} \circ s : [n] \to [n]$ is merely a permutation, not necessarily an identity. Intuitively, symmetries permute the numbers that label interfaces. The following lemma explains their name.

Lemma 8. *For any $n \geq 0$, the automorphisms on $[n]$ are precisely the symmetries $n \to n$ in* iPos.

Proof. Any symmetry $n \to n$ is obviously an isomorphism on $[n]$. For the converse, let $(s, P, t) : n \to n$ be an isomorphism and $(s_1, Q_1, t_1) : n \to n$ and $(s_2, Q_2, t_2) : n \to n$ its left and right inverses. The numbers of points in Q_1 and P satisfy $|Q_1| \geq |T_{Q_1}| = n$ and $|P| \geq |S_P| = n$, and then $n = |Q_1 \triangleright P| = |Q_1| + |P| - n$ implies $|P| = n$, so that P must be a terminator. A symmetric argument using Q_2 shows that P is a starter, hence a symmetry. □

The next lemma shows that parallel composition of iposets commutes up to symmetry.

Lemma 9. *For any iposets $P_1 : n_1 \to m_1$ and $P_2 : n_2 \to m_2$ there are symmetries σ on $n_1 + n_2$ and τ on $m_1 + m_2$ such that $P_1 \otimes P_2 \cong \sigma \rhd (P_2 \otimes P_1) \rhd \tau$.*

Proof. In light of Lemma 8, the symmetries may be defined by

$$\sigma(i) = \begin{cases} i + n_2 & \text{for } i \leq n_1, \\ i - n_1 & \text{for } i > n_1 \end{cases} \qquad \text{and} \qquad \tau(i) = \begin{cases} i + m_1 & \text{for } i \leq m_2, \\ i - m_2 & \text{for } i > m_2. \end{cases} \quad \square$$

Non-identity symmetries may be removed from our setting by imposing interface consistency, which we define next.

The interfaces of an iposet $(s, P, t) : n \to m$ induce implicit extra orderings on some of the points of P that are independent of the order on P. They are defined by $x \dashrightarrow_s y$ if $x, y \in S_P$ and $s^{-1}(x) <_{\mathbb{N}} s^{-1}(y)$, and $x \dashrightarrow_t y$ if $x, y \in T_P$ and $t^{-1}(x) <_{\mathbb{N}} t^{-1}(y)$. Here $<_{\mathbb{N}}$ is the natural ordering on $[n]$ and $[m]$.

Definition 9. Iposet $(s, P, t) : n \to m$ is *interface consistent* if

$$s^{-1}(x) <_{\mathbb{N}} s^{-1}(y) \Leftrightarrow t^{-1}(x) <_{\mathbb{N}} t^{-1}(y) \qquad \text{for all } x, y \in S_P \cap T_P.$$

The orders $\dashrightarrow_s$ and $\dashrightarrow_t$ of interface consistent iposets can therefore be combined into a partial order $\dashrightarrow \,=\, \dashrightarrow_s \cup \dashrightarrow_t$ on P. The interface consistent symmetries are precisely the identities, as all points of a symmetry are in $S_P \cap T_P$. Further, interface consistency is preserved by gluing composition, parallel composition and subsumption. Interface consistent iposets thus form a subcategory of iPos. By Lemma 8, all its automorphisms are identities.

We will see in Lemma 13 that all iposets generated from singletons using finitary gluing and parallel compositions are interface consistent. On the other hand, there are interface consistent iposets which are not gluing-parallel, see Example 3 below.

7. A Criterion for Gluing Decompositions

In this section we supply a criterion for the existence of gluing decompositions in iposets. When thinking of a decomposition $P = Q \rhd R$ as synchronic cut through P, then some events are in the past, already terminated, some in the present, currently running, and some in the future, yet to be started. This is captured in the following definition.

Definition 10. Let $Q : n \to m$ and $R : m \to k$ be iposets. The *characteristic function* of the decomposition $Q \rhd R$ is $\phi_{Q \rhd R} : Q \rhd R \to \{0, *, 1\}$ defined by

$$\phi_{Q \rhd R}(x) = \begin{cases} 1 & \text{for } x \in Q \setminus T_Q, \\ * & \text{for } x \in T_Q = S_R, \\ 0 & \text{for } x \in R \setminus S_R. \end{cases}$$

Hence we label past events with respect to the decomposition with 1, present events with $*$, and future events with 0.

Lemma 10. *The characteristic function $\phi = \phi_{Q \triangleright R}$ satisfies the following:*

(A) If $(\phi(x), \phi(y)) = (1, 0)$, then $x < y$.

(B) If $(\phi(x), \phi(y)) \in \{(1, *), (*, 0), (1, 0)\}$, then $y \not< x$.

(C) If $(\phi(x), \phi(y)) = (*, *)$, then $x \not< y$ and $y \not< x$.

(D) If $x < y$ and $\phi(y) \neq 0$, then $\phi(x) = 1$. If $x < y$ and $\phi(x) \neq 1$, then $\phi(y) = 0$.

If the decomposition $P = Q \triangleright R$ is non-trivial, then there exist $x, y \in P$ such that x is minimal, y is maximal, $\phi(x) = 1$, and $\phi(y) = 0$.

Proof. (A) follows immediately from the definition of $\triangleright$. (B) holds because source (target) interfaces contain only minimal (maximal) elements and by (A). (C) follows from incomparability of interface elements. (D) follows from (B) and (C). The last statement is obvious. $\square$

Let P be a poset, and let $P_a, P_b \subseteq P$ be the sets of points x for which the up-sets $x{\uparrow} = \{y \mid x < y\}$ and down-sets $x{\downarrow} = \{y \mid y < x\}$ have maximal size,

$$P_a = \big\{x \in P \mid \forall y \in P.\ |y{\uparrow}| \leq |x{\uparrow}|\big\}, \qquad P_b = \big\{x \in P \mid \forall y \in P.\ |y{\downarrow}| \leq |x{\downarrow}|\big\}.$$

Then $P_a \neq \emptyset \neq P_b$, every element of P_a is minimal in P and every element of P_b is maximal in P.

Lemma 11. *Suppose P admits a non-trivial gluing decomposition. Then there is $\phi : P \to \{0, *, 1\}$ that satisfies the conditions of Lemma 10 and, in addition,*

(E) $\phi(x) = 1$ for $x \in P_a$ and $\phi(y) = 0$ for $y \in P_b$.

Proof. We only verify (E). Suppose $P = Q \triangleright R$ is non-trivial. Then there must be $y, z \in P$ such that $\phi_{Q \triangleright R}(y) = 1$, $\phi_{Q \triangleright R}(z) = 0$ and therefore $y < z$.

Let $x \in P_a$. If $\phi_{Q \triangleright R}(x) = 0$, then $x > y$ by (A), which contradicts minimality of x. Suppose $\phi_{Q \triangleright R}(x) = *$, then $x{\uparrow} \subseteq R \setminus S_R \subseteq y{\uparrow}$. But $|x{\uparrow}| \geq |y{\uparrow}|$, forcing $x{\uparrow} = y{\uparrow}$. In other words, $x < w$ for any $w \in R \setminus S_R$. Hence there is a non-trivial gluing decomposition $P = Q' \triangleright R'$ in which x has been moved from the gluing interface into $Q' \setminus T_{Q'}$ and consequently $\phi_{Q' \triangleright R'}(x) = 1$.

The proof of $\phi_{Q \triangleright R}(x) = 0$ for $x \in P_b$ follows by opposition. $\square$

The following is now immediate from (A) and (E).

Corollary 12. *If poset P admits a non-trivial gluing decomposition, then $x < y$ for all $x \in P_a$ and $y \in P_b$.* $\square$

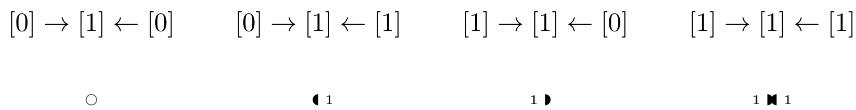


Figure 9: The four singletons, structurally and graphically.

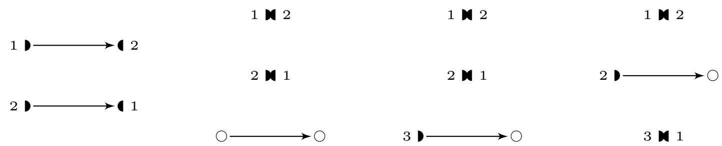


Figure 10: Four iposets on four points that are not gluing-parallel.

8. Gluing-Parallel Iposets

In this section, we start our study of iposets generated from singletons using $\triangleright$ and $\otimes$. A *singleton* is an iposet whose underlying poset has one single point. There are four of them: $\circ$, ◖$_1$, $_1$◗ and $_1$⧓$_1$. In particular, $\mathsf{id}_1 =$ $_1$⧓$_1$. Figure 9 shows their structural definitions and graphical representations. We write $\mathcal{S} = \{\circ,$ ◖$_1$, $_1$◗, $_1$⧓$_1\}$ for the set of singleton iposets. For $i, j \in \{0, 1\}$, we write more generally $_i \circ _j$ instead of $(f_i, [1], f_j) : n \to m$, where $f_k : [k] \to [1]$ is either empty or the identity.

Definition 11. The set of *gluing-parallel* iposets (*gp-iposets*) is the smallest set that contains the empty iposet id_0 and all elements of $\mathcal{S}$ and is closed under gluing and parallel composition.

Lemma 13. *Every gp-iposet is interface consistent.*

Proof. The empty iposet and all singletons are interface consistent; gluing and parallel compositions preserve this property. □

Example 3. Interface consistency characterises gp-iposets on two and three points: on two points, the only iposet that is not in this class is the non-trivial symmetry on [2]; on three points, twelve iposets are not in it. All of them are discrete and not interface consistent.

On four points, there are 113 non-gp iposets; 96 of them are discrete (and not interface consistent). Of the seventeen others, sixteen are parallel products of the non-trivial symmetry on [2] with an arrow. The last one is the 2+2 with interfaces swapped, which is the only interface consistent iposet on four points which is not gluing-parallel. In Figure 10, the latter is displayed on the left, followed by three examples of the former.

Recall that the *series-parallel* posets are freely generated from the empty poset and the singleton $\circ$ by finitary serial and parallel compositions and up to unicity and associativity of these compositions as well as commutativity of parallel product [1]. There remain several obstructions to showing a similar result for gp-iposets.

First, the lax interchange law from Proposition 2 becomes strict if two of the components involved are singletons with matching interfaces.

Proposition 14 (Singleton interchange). *For all iposets P, Q with $P \triangleright Q$ defined and $i, j \in \{0, 1\}$,*

$$({}_i\!\circ_1 \otimes P) \triangleright ({}_1\!\circ_j \otimes Q) \cong ({}_i\!\circ_1 \triangleright {}_1\!\circ_j) \otimes (P \triangleright Q),$$
$$(P \otimes {}_i\!\circ_1) \triangleright (Q \otimes {}_1\!\circ_j) \cong (P \triangleright Q) \otimes ({}_i\!\circ_1 \triangleright {}_1\!\circ_j).$$

Proof. As ${}_i\!\circ_1$ and ${}_1\!\circ_j$ are both single points in an interface, the result follows by definition of $\triangleright$: the singletons glue separately in the left-hand sides of the identities precisely in the way described in their right-hand sides.

For a detailed proof, first note that ${}_i\!\circ_1 \triangleright {}_1\!\circ_j = {}_i\!\circ_j$. By Proposition 2, there is a subsumption $f : ({}_i\!\circ_1 \otimes P) \triangleright ({}_1\!\circ_j \otimes Q) \to {}_i\!\circ_j \otimes (P \triangleright Q) = L \to R$, which is the identity on the two underlying posets. It remains to show that it preserves the order. So let $x, y \in L$ with $x <_L y$. If $x, y \in {}_i\!\circ_1 \otimes P$, then $x <_P y$, hence $x <_{P \triangleright Q} y$ and $x <_R y$, and likewise for $x, y \in {}_1\!\circ_j \otimes Q$. Otherwise, if $x \in {}_i\!\circ_1 \otimes P \setminus T_{{}_i\circ_1 \otimes P}$ and $y \in {}_1\!\circ_j \otimes Q \setminus S_{{}_1\circ_j \otimes Q}$, then $x \in P \setminus T_P$ and $y \in Q \setminus S_Q$, thus $x <_{P \triangleright Q} y$ and again $x <_R y$. □

Second, parallel composition becomes commutative when some components have no interfaces.

Proposition 15. *Let $P_1 : n_1 \to m_1$ and $P_2 : n_2 \to m_2$ be iposets and assume that $n_1 = 0$ or $n_2 = 0$, and $m_1 = 0$ or $m_2 = 0$. Then $P_1 \otimes P_2 \cong P_2 \otimes P_1$.*

Proof. Using Lemma 9 there are symmetries σ and τ such that $P_1 \otimes P_2 \cong \sigma \triangleright (P_2 \otimes P_1) \triangleright \tau$; but the assumptions make both σ and τ identities. □

Propositions 14 and 15 have converses, which tell us precisely when strict interchange and commutativity hold. First, by the next lemma, Proposition 14 covers all cases of strict interchange.

Lemma 16. *Let $P_1 \otimes P_2 \cong Q_1 \triangleright Q_2$ such that the gluing composition is non-trivial. Then P_1 or P_2 is discrete.*

Proof. Partition each P_i into P_{ij} containing the points in P_i that are also in Q_j, as indicated in Figure 11. By hypothesis, there is a non-target point $p \in Q_1 \setminus T_{Q_1}$ and a non-source point $q \in Q_2 \setminus S_{Q_2}$, hence $p < q$ in $Q_1 \triangleright Q_2$. More precisely, we must have $p \in P_{i1}$ and $q \in P_{i2}$ for i either 1 or 2, because P_1 and P_2 are $<$-disconnected. Now any non-terminating point $r \in P_{j1}$ for $j \neq i$ would force $r < q$ in $Q_1 \triangleright Q_2$, which is inconsistent with this disconnectivity. By opposition, any non-starting point $r \in P_{j2}$ would force $p < r$ in $Q_1 \triangleright Q_2$. Hence P_{j1} must be a starter and P_{j2} a terminator, making P_j discrete. □

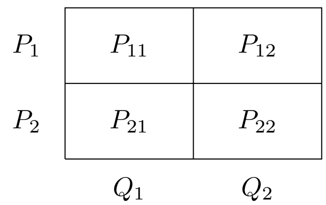


Figure 11: Partition of $P_1 \otimes P_2 \cong Q_1 \triangleright Q_2$ in the proof of Lemma 16.

Let $\sim$ denote the equivalence relation generated by $<$ in a poset P. Equivalence classes of $\sim$ are *connected components* of P. P is *connected* if it has exactly one connected component.

Lemma 17. *Let $P_1 : n_1 \to m_1$ and $P_2 : n_2 \to m_2$ be iposets such that P_1 and P_2 are both connected, $P_1 \not\cong P_2$, and $P_1 \otimes P_2 \cong P_2 \otimes P_1$. Then $n_1 = 0$ or $n_2 = 0$, and $m_1 = 0$ or $m_2 = 0$.*

Proof. Let $f : P_1 \otimes P_2 \to P_2 \otimes P_1$ be any iposet isomorphism and suppose $n_1 \neq 0 \neq n_2$. We show that $f|_{P_1} : P_1 \to P_2$ is an isomorphism, too. The proof for $m_1 \neq 0 \neq m_2$ is then symmetric.

Both $P_1 \otimes P_2$ and $P_2 \otimes P_1$ have two connected components: $[P_1]$ and $[P_2]$. The condition $n_1 \neq 0 \neq n_2$ guarantees that $f(s_{P_1 \otimes P_2}(1)) = s_{P_2 \otimes P_1}(1) \in P_2$ and $s_{P_1 \otimes P_2}(1) \in P_1$. Thus, f sends the connected component $[P_1] \subseteq P_1 \otimes P_2$ to $[P_2] \subseteq P_2 \otimes P_1$. Since f is an isomorphism, its restriction $f|_{P_1} : P_1 \to P_2$ is also an isomorphisms of posets.

Let $s_{12} : [n_1 + n_2] \to P_1 \otimes P_2$, $s_{21} : [n_1 + n_2] \to P_2 \otimes P_1$ denote the respective source interfaces for short. By assumption, $f \circ s_{12} = s_{21}$. Thus, for $i \in [n_1]$,

$$f(s_1(i)) = f((s_1 \otimes s_2) \circ \varphi_{n_1,n_2}(i)) = f(s_{12}(i)) = s_{21}(i) = (s_2 \otimes s_1) \circ \varphi_{n_2,n_1}(i).$$

We have $s_1(i) \in [P_1] \subseteq P_1 \otimes P_2$ and, therefore, $f(s_1(i)) \in P_2 \subseteq [P_2]$. Hence, $\varphi_{n_2,n_1}(i) \leq n_2$, *i.e.*, $i \leq n_2$. Finally,

$$f(s_1(i)) = (s_2 \otimes s_1) \circ \varphi_{n_2,n_1}(i) = s_2(i).$$

Similarly we show that $f \circ t_1 = t_2$. This implies that $f|_{P_1} : P_1 \to P_2$ preserves interfaces and is an isomorphism of iposets. $\square$

The connectedness assumption above is needed for rather trivial reasons: if $P \neq \mathsf{id}_0$, then $P \neq P \otimes P$, but $P \otimes (P \otimes P) = (P \otimes P) \otimes P$.

Whether gluing-parallel iposets form the free algebra in some variety remains open. Such a result is claimed in [9, Thm. 19], but its proof depends on a lemma, [9, Lemma 16], that does not hold. It states that if $P \triangleright Q = U \triangleright V$ for some iposets, then these gluing decompositions have a common refinement, that is, there exists an iposet R such that either $P = U \triangleright R$ and $R \triangleright Q = V$ or $U = P \triangleright R$ and $R \triangleright V = Q$. See Figure 12 for an illustration. This claim is refuted by the following simple counterexample.

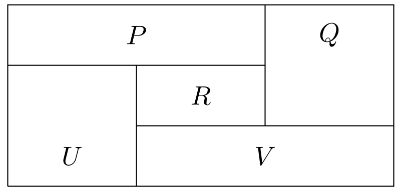
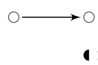
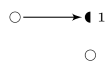
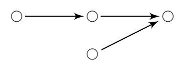


Figure 12: One of the two situations in Levi's lemma.

Example 4. Consider the iposets

$P =$  , $\quad U =$  , $\quad V =$.

Then

$P \rhd V = U \rhd V =$  ,

but neither $P \rhd R = U$ nor $U \rhd R = P$ for any iposet R.

9. A Hierarchy of Gluing-Parallel Iposets

In order to refine our study of gluing-parallel iposets, and in particular to work towards a result on free generation as mentioned above, we now define a hierarchy of gp-iposets generated from the empty iposet and the singletons in $\mathcal{S}$ by successive iterations of parallel and gluing compositions. We show that this hierarchy is infinite, which is a necessary, but not a sufficient condition for gp-iposets being freely generated.

For any $\mathcal{Q} \subseteq \mathsf{iPos}$ and $\square \in \{\otimes, \rhd\}$ let

$$\square\mathcal{Q} = \{P_1 \square \cdots \square P_n \mid n \geq 1, P_1, \ldots, P_n \in \mathcal{Q}\}$$

be the $\square$-closure of $\mathcal{Q}$. In particular, therefore, $\mathcal{Q} \subseteq \square\mathcal{Q}$. This defines operations $\rhd, \otimes : 2^{\mathsf{iPos}} \to 2^{\mathsf{iPos}}$. We are interested in their iterative application and write

$$\mathcal{S}_0 = \{\mathsf{id}_0\} \cup \mathcal{S}, \qquad \mathcal{S}_{i+1} = \rhd\otimes\mathcal{S}_i, \qquad \mathcal{S}_* = \bigcup_{i \geq 0} \mathcal{S}_i.$$

Then $\mathcal{S}_*$ is the class of gp-iposets.

Further, we are interested in gluing-parallel *posets*, which we regard as iposets with empty interfaces. To this end, let

$$\mathcal{G} = \mathcal{S} \cap \mathsf{Pos}, \qquad \mathcal{G}_n = \mathcal{S}_n \cap \mathsf{Pos}, \qquad \mathcal{G}_* = \mathcal{S}_* \cap \mathsf{Pos}.$$

Lemma 18. *An iposet is in $\otimes\mathcal{S}$ if and only if it is discrete and interface consistent.*

Proof. Any iposet in $\otimes\mathcal{S}$ is obviously discrete and interface consistent. Conversely, let $(s, P, t) : n \to m$ be discrete and interface consistent. We extend the partial order $\dashrightarrow$ defined after Definition 9 to a linear order, also denoted $\dashrightarrow$, on P. If the components of P are ordered so that $P_1 \dashrightarrow \cdots \dashrightarrow P_k$, then $P = P_1 \otimes \cdots \otimes P_k \in \otimes\mathcal{S}$, with $P_i \in \mathcal{S}$ for $1 \le i \le k$. □

Lemma 19. *If $(s, P, t) : n \to m$ is in $\mathcal{S}_k$, then so is $(\emptyset, P, \emptyset) : 0 \to 0$, and $\mathcal{G}_k$ is the set of underlying posets of iposets in $\mathcal{S}_k$.*

Proof. This is straightforward for $k = 0$. For $k > 0$, there is a presentation $(\emptyset, P, \emptyset) = S \triangleright (s, P, t) \triangleright T$, where $S : 0 \to n$ is a starter and $T : m \to 0$ is a terminator. Since $S, T \in \otimes\mathcal{S} \subseteq \mathcal{S}_1$, the claim follows. □

Next we show that the interval orders are precisely the posets in $\mathcal{G}_1$. We use the following technical lemma, which uses a suitable finite linear order as the image of an interval representation instead of $\mathbb{R}$.

Lemma 20. *Every iposet (s, P, t) with P an interval order has an interval representation $(b, e) : P \to Q$ into a finite linear order Q such that $x \in S_P$ if and only if $b(x)$ is the least element $\bot$ of Q and $x \in T_P$ if and only if $e(x)$ is the greatest element $\top$ of Q. Moreover, we may assume that $|Q| \ge 3$ and $b(p) < \top$, $e(p) > \bot$ for all $p \in P$.*

Proof. For any interval presentation $(b', e') : P \to Q'$, let Q be the linear order obtained by adjoining a new minimal element $\bot$ and a new maximal element $\top$ to Q'. Then

$$b(x) = \begin{cases} b'(x) & \text{for } x \notin S_P, \\ \bot & \text{for } x \in S_P \end{cases} \qquad \text{and} \qquad e(x) = \begin{cases} e'(x) & \text{for } x \notin T_P, \\ \top & \text{for } x \in T_P \end{cases}$$

define an interval presentation $(b, e) : P \to Q$ satisfying the claim. □

Proposition 21. *The class $\mathcal{S}_1$ is equal to the class of interface consistent interval orders. The class $\mathcal{G}_1$ is equal to the class of interval orders.*

Proof. It suffices to prove the first claim; the second one follows. The underlying posets of all elements of $\otimes\mathcal{S}$ are interval orders. To prove the forward implication, it thus suffices to show that the gluing of two interval orders yields an interval order. Suppose $(b_P, e_P) : P \to R$, $(b_Q, e_Q) : Q \to S$ are interval representations satisfying the conditions of Lemma 20. Then

$$b(x) = \begin{cases} b_P(x) & \text{for } x \in P, \\ b_Q(x) & \text{for } x \notin P \end{cases} \qquad \text{and} \qquad e(x) = \begin{cases} e_P(x) & \text{for } x \notin Q, \\ e_Q(x) & \text{for } x \in Q \end{cases}$$

define an interval representation $P \triangleright Q \to (R \sqcup S)/_{\top_R = \bot_S}$. Figure 13 shows an example.

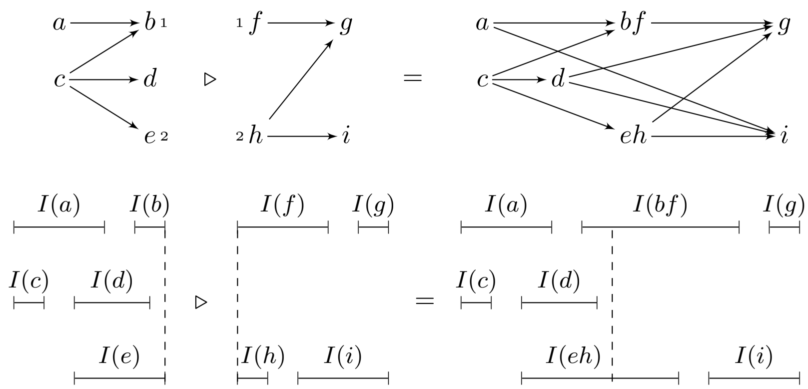


Figure 13: Two interval orders and their concatenation: above as iposets, below using their interval representations. (Labels added for convenience.)

Conversely, suppose iposet (s, P, t) is an interface consistent interval order with an interval representation $b, e : P \to Q = \{\bot < q_1 < \cdots < q_n < \top\}$ satisfying the conditions of Lemma 20. We prove $P \in \mathcal{S}_1$ by induction on n. If $n = 1$, then $b(p) \le q_1 \le e(p')$ for all $p, p' \in P$. Thus, P is discrete and interface consistent and by Lemma 18 belongs to $\otimes\mathcal{S} \subseteq \mathcal{S}_1$.

Assume $n > 1$. Extend the order $\dashrightarrow$ introduced below Definition 9 to an arbitrary linear order, also denoted $\dashrightarrow$. Let $P' = \{p \in P \mid b(p) \le q_1\}$, $P'' = \{p \in P \mid e(p) \ge q_2\}$ be induced subposets of P. The intersection $P' \cap P''$ is a discrete poset. Let $m = |P' \cap P''|$ and $u : [m] \to (P' \cap P'', \dashrightarrow)$ be the unique order-preserving function. Both iposets (s, P', u) and (u, P'', t) are interface consistent interval iposets, and both have shorter interval representations than P, namely, $b', e' : P' \to \{\bot < q_1 < \top\}$ and $b'', e'' : P'' \to \{\bot < q_2 < \cdots < q_n < \top\}$ given by $b'(p) = b(p)$, $e''(p) = e(p)$ and

$$e'(p) = \begin{cases} e(p) & \text{for } e(p) \le q_1 \\ \top & \text{for } e(p) > q_1 \end{cases}, \qquad b''(p) = \begin{cases} b(p) & \text{for } b(p) \le q_2 \\ \bot & \text{for } b(p) > q_2 \end{cases}.$$

Hence, by inductive hypothesis, $P', P'' \in \mathcal{S}_1$. It is elementary to verify that $(s, P, t) = (s, P', u) \triangleright (u, P'', t)$, which implies that $(s, P, t) \in \mathcal{S}_1$. $\square$

To compare gluing-parallel posets with series-parallel ones we construct a similar hierarchy for them. Let $\mathcal{T}_0 = \mathcal{G}_0 = \{\mathsf{id}_0, \circ\} \subset \mathcal{S}_0$ be the set containing the empty poset and the unique singleton without interfaces and, like for the $\mathcal{S}$ hierarchy above, $\mathcal{T}_{n+1} = \triangleright\otimes\mathcal{T}_n$, and $\mathcal{T}_* = \bigcup_{n \ge 0} \mathcal{T}_n$.

Lemma 22. *A poset is series-parallel if and only if it is in $\mathcal{T}_*$.*

Proof. The elements of $\mathcal{T}_0$ are the empty poset and the singleton with empty interfaces, and if every element of any $\mathcal{Q} \subseteq \mathsf{iPos}$ has empty interfaces, then the same holds for $\otimes\mathcal{Q}$ and $\triangleright\mathcal{Q}$. Thus any element of any $\mathcal{T}_n$ has empty interfaces.

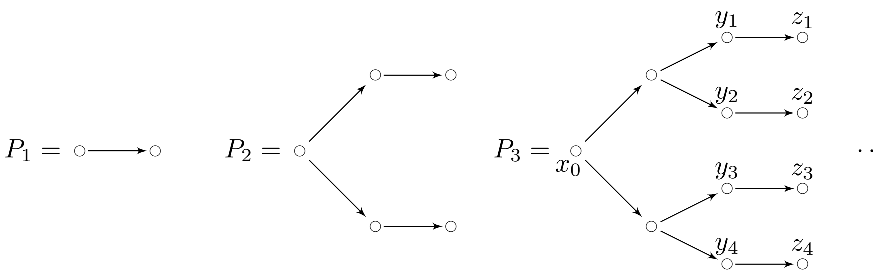


Figure 14: Sequence of separators for $\mathcal{T}_n$ and $\mathcal{S}_{n-1}$.

Further, if P and Q have empty interfaces, then $P \triangleright Q$ is a serial composition. The claim then follows. $\square$

Lemma 23. *$\mathcal{T}_n \subset \mathcal{G}_n$ for all $n \geq 1$, and $\mathcal{T}_* \subset \mathcal{G}_*$.*

Proof. $\mathcal{T}_0 \subset \mathcal{S}_0$ implies $\mathcal{T}_n \subseteq \mathcal{S}_n$. Hence $\mathcal{T}_n = \mathcal{T}_n \cap \mathsf{Pos} \subseteq \mathcal{S}_n \cap \mathsf{Pos} = \mathcal{G}_n$, for all $n \geq 0$. Thus also $\mathcal{T}_* \subseteq \mathcal{G}_*$, and the inequalities follow from $\mathcal{T}_* \not\ni \mathsf{N} \in \mathcal{G}_1$. $\square$

For an analogue of Proposition 21 for sp-posets, recall [32] that a poset P is a *step sequence* if its incomparability relation $\|$ defined by $x \parallel y \Leftrightarrow x \not< y \wedge y \not< x$ is transitive, so that every point belongs to a unique $\|$-equivalence class.

Proposition 24. *A poset is in $\mathcal{T}_1$ if and only if it is a step sequence.*

Proof. If $P = P_1 \triangleright \cdots \triangleright P_n$ for $P_1, \dots, P_n \in \otimes\mathcal{T}_0$, then each P_i is discrete, hence $\{P_1, \dots, P_n\}$ is the $\|$-partition of P. Conversely, if $\|$ is transitive, then the $\|$-partition of P can be totally ordered as $P_1 < \cdots < P_n$, and then $P = P_1 \triangleright \cdots \triangleright P_n$. $\square$

Next we show that our three hierarchies are infinite, presenting a sequence of witnesses for the strictness of inclusions. Let $P_1 = \circ \triangleright \circ$, and for $n \geq 1$, $P_{n+1} = \circ \triangleright (P_n \otimes P_n)$. See Figure 14 for a graphical representation. Note that all of them are series-parallel posets.

Lemma 25. *Let $P_{n+1} = Q \triangleright R$ be a non-trivial gluing decomposition for $n \geq 1$. Then $R = P_n \otimes P_n$ as posets.*

Proof. Set $P = P_{n+1}$ and let $\phi = \phi_{Q \triangleright R}$ be the characteristic function of $Q \triangleright R$ (see Definition 10). Then $P_a = \{x_0\}$ is the set of points for which the up-sets have maximal size because x_0 is the only minimal element of P, and $P_b = \{z_i \mid i \in [2^n]\}$ is the set of points for which the down-sets have maximal size, because $|z_i{\downarrow}| = n + 1$ for all i.

Lemma 10 implies that $\phi(x_0) = 1$ and $\phi(z_i) = 0$ for some i. For any $j \in [2^n] \setminus \{i\}$, $y_j \not< z_i$ and therefore $\phi(y_j) \neq 1$ by Lemma 10(A). Then $\phi(z_j) = 0$ because $y_j < z_j$, using Lemma 10(D). Now if $\phi(p) = 1$, then $p < z_j$ for all j, which implies that $p = x_0$. Finally,

$$R = \phi^{-1}(\{*, 0\}) = P_{n+1} \setminus \{x_0\} = P_n \otimes P_n. \qquad \square$$

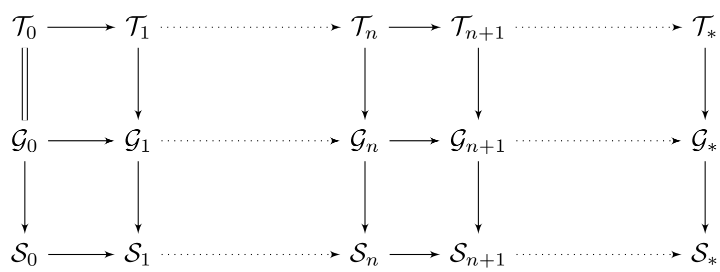


Figure 15: Hasse diagram of equalities and proper inclusions between sp-posets (top row), gp-posets (middle) and gp-iposets (bottom row).

Lemma 26. *$P_n \in \mathcal{T}_n \setminus \mathcal{G}_{n-1}$ for all $n \geq 1$.*

Proof. A simple induction shows that $P_n \in \mathcal{T}_n$ for $n > 1$. It remains to show that $P_n \notin \mathcal{G}_{n-1}$ for all $n > 1$. We proceed again by induction. In the base case, $P_1 \notin \mathcal{G}_0$. For the induction step, suppose $P_n \notin \mathcal{G}_{n-1}$. We need to show that $P_{n+1} \notin \mathcal{G}_n$. As P_{n+1} is connected, it is of the form $Q \triangleright R$, and non-trivially so. Hence, by Lemma 25, $R = P_n \otimes P_n$. We know that $P_n \notin \mathcal{G}_{n-1}$. It remains to show that $P_n \otimes P_n \notin \mathcal{G}_{n-1}$. This follows from Corollary 12, as P_a consists of the two minimal elements of $P_n \otimes P_n$ and P_b of all its maximal elements. Nevertheless there are no arrows from one copy of P_n into the other. Hence $P_n \otimes P_n$ can only be written as a parallel composition of P_n with itself. □

Corollary 27. *$\mathcal{G}_n \subset \mathcal{G}_{n+1}$, $\mathcal{T}_n \subset \mathcal{T}_{n+1}$ and $\mathcal{T}_* \not\subseteq \mathcal{G}_n$ for any $n \geq 0$.*

Proof. $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ and $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$ by definition, and by Lemma 26,

$$\mathcal{G}_{n+1} \setminus \mathcal{G}_n \cap \mathcal{T}_{n+1} \setminus \mathcal{T}_n = \mathcal{T}_{n+1} \setminus \mathcal{G}_n \supseteq \{P_{n+1}\} \neq \emptyset$$

for all n. The last claim follows from the fact that $P_n \in \mathcal{T}_*$ for all n. □

We summarise the relationships between the different sets in the next proposition, see Figure 15 for an illustration.

Proposition 28. *For all $n \geq 0$, $\mathcal{T}_n \subset \mathcal{T}_{n+1}$, $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ and $\mathcal{S}_n \subset \mathcal{S}_{n+1}$. For all $n \geq 1$, $\mathcal{T}_n \subset \mathcal{G}_n \subset \mathcal{S}_n$ and $\mathcal{T}_0 = \mathcal{G}_0 \subset \mathcal{S}_0$. For all $n, m \geq 1$, $\mathcal{G}_n \not\subseteq \mathcal{T}_m$ and $\mathcal{S}_n \not\subseteq \mathcal{G}_m$.*

Proof. We have already shown all but the third and the last two claims, and $\mathcal{S}_n \subset \mathcal{S}_{n+1}$ follows from the definition $\mathcal{G}_n = \mathcal{S}_n \cap \mathsf{Pos}$. Now if $\mathcal{G}_n \subseteq \mathcal{T}_m$ for some $n, m \geq 1$, then also $\mathcal{G}_n \subseteq \mathcal{T}_*$ in contradiction to $\mathcal{T}_* \not\ni \mathsf{N} \in \mathcal{G}_1$. This also shows that $\mathcal{T}_n \subset \mathcal{G}_n$ for $n \geq 1$. The last claim is clear as $\mathcal{S}_0$ contains iposets with nonempty interfaces. □

10. Forbidden Substructures

In this section we collect some combinatorial properties of gluing-parallel posets and iposets and expose eleven forbidden substructures. We do not know whether our list of forbidden substructures is comprehensive.

It is well-known, and follows directly from their characterisation as being N-free, that series-parallel posets are closed under induced subposets. We show that the same is true for gp-posets.

Proposition 29. *All induced subposets of gp-posets are gluing-parallel.*

Proof. We prove by induction that removing one point from a gp-poset yields a gp-poset. The claim then follows by another induction.

Our property is obviously true for the singleton poset. Now let P be gluing-parallel and $x \in P$. If P has a non-trivial parallel decomposition $P = Q \otimes R$, then either $x \in Q$ or $x \in R$, and these cases are symmetric. So suppose $x \in Q$. Then $Q \setminus \{x\}$ is gluing-parallel by the inductive hypothesis and $P \setminus \{x\} = (Q \setminus \{x\}) \otimes R$.

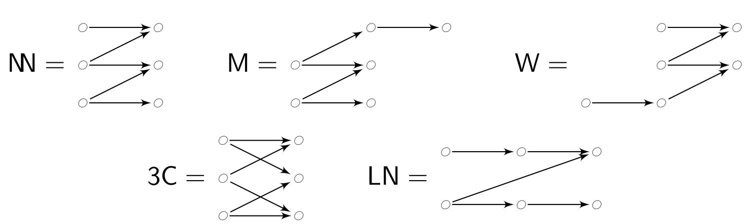
If P has a non-trivial gluing decomposition $P = Q \triangleright R$, then either $x \in Q \setminus T_Q$, $x \in R \setminus S_R$, or $x \in T_Q = S_R$. The first two cases can be handled just as above, given that $P \setminus \{x\} = (Q \setminus \{x\}) \triangleright R$ respectively $P \setminus \{x\} = Q \triangleright (R \setminus \{x\})$. For the last case, $P \setminus \{x\} = (Q \setminus \{x\}) \triangleright (R \setminus \{x\})$. □

By collecting all posets that are not gluing-parallel and weeding out induced subposets, we obtain the following consequence of Proposition 29.

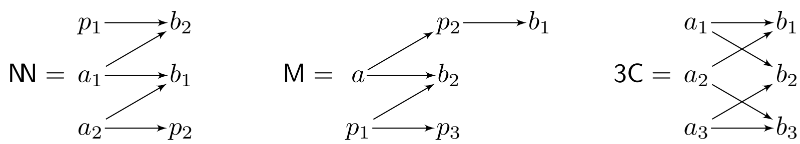
Corollary 30. *There exists a set $\mathcal{F}$ of posets such that for any $P, Q \in \mathcal{F}$ with $P \neq Q$, neither P nor Q is an induced subposet of the other, and such that any poset is gluing-parallel if and only if it does not contain any element of $\mathcal{F}$ as an induced subposet.*

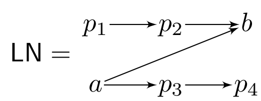
Such a set $\mathcal{F}$ is said to consist of *forbidden substructures*. We have already mentioned that sp-posets have precisely one forbidden substructure, the poset N; also interval orders have precisely one forbidden substructure, the poset 2+2. The question is now whether gp-posets admit a *finite* set of forbidden substructures.

Proposition 31. *The following five posets are forbidden substructures for gp-posets:*



Proof. W is the opposite of M, so we can ignore it for this proof. We use the notation introduced below Lemma 10 and label the vertices of the remaining four posets so that elements of P_a are labelled a_i, elements of P_b are labelled b_i, and the remaining elements are labelled p_i:





Let $P \in \{\mathsf{NN}, \mathsf{M}, \mathsf{3C}, \mathsf{LN}\}$ and assume that P has a non-trivial gluing decomposition. Lemma 11 implies that there exists a function $\phi : P \to \{0, *, 1\}$ satisfying Lemma 10 such that $\phi(a_i) = 1$ and $\phi(b_i) = 0$.

For $P = \mathsf{NN}$ and $P = \mathsf{3C}$ we have $a_2 \not< b_2$ in contradiction to $\phi(a_2) = 1$ and $\phi(b_2) = 0$. For $P = \mathsf{M}$, $a \not< p_1 \not< b_1$ implies $\phi(p_1) \notin \{0, 1\}$ by the same argument, thus $\phi(p_1) = *$. Similarly, $a \not< p_3 \not< b_1$ implies $\phi(p_3) = *$. This contradicts $p_1 < p_3$ by Lemma 10(C).

For $P = \mathsf{LN}$, $a \not< p_2$ implies $\phi(p_2) \neq 0$, *i.e.*, $\phi(p_2) \in \{*, 1\}$, and then $p_1 < p_2$ implies $\phi(p_1) = 1$ by Lemma 10(D). Dually, $p_3 \not< b$ implies $\phi(p_3) \neq 1$ and then $\phi(p_4) = 0$. But $p_1 \not< p_4$, a contradiction.

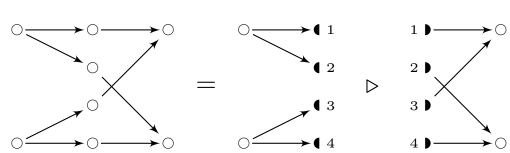
The proof that all proper subposets of these posets are gluing-parallel is trivial verification. □

In addition to the above five posets, we have found six other elements of $\mathcal{F}$. Whether $\mathcal{F}$ is finite or infinite is open.

Proposition 32. *The six posets in Figure 16 are forbidden substructures for gp-posets.*

Proof. By computer (see below).

Remark 4. The 8-point forbidden substructure in the centre of the top row of Figure 16 has a gluing decomposition along a maximal antichain:



Now the first of the iposets on the right-hand side is easily seen to be gluing-parallel, given that it is the parallel product of two three-point iposets. The second iposet, however, has its interfaces swapped so that it is *not* a parallel product; it is not gluing-parallel.

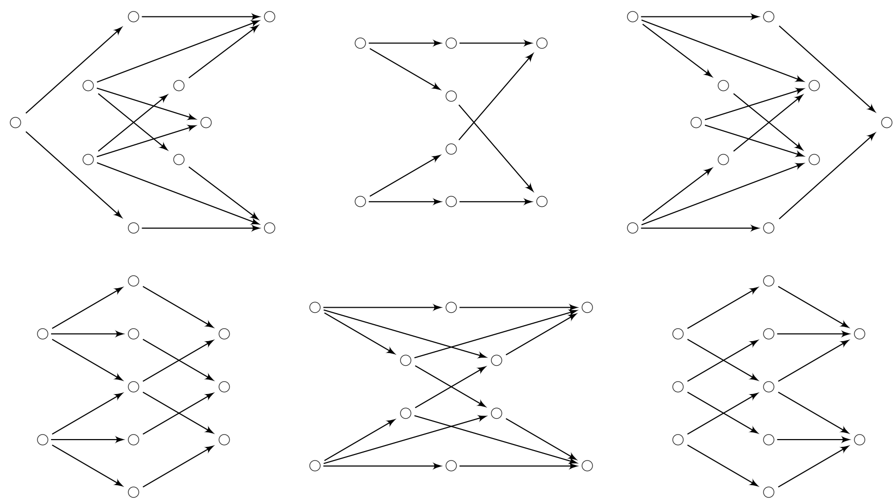


Figure 16: Additional forbidden substructures for gp-posets.

We have written a Julia program, using the LightGraphs package [5], to generate gluing-parallel iposets and analyse their properties.[1] The next proposition is a result of these calculations.

Proposition 33. *The eleven posets of Propositions 31 and 32 are the only posets in $\mathcal{F}$ with at most 10 points.* □

In other words, any further forbidden substructures must have at least 11 points. Generating posets is notoriously difficult [4], so any improvements to the above results are left for future work. We have also used our software to count non-isomorphic posets and iposets of different types, see Table 1. As a refutation of a conjecture in [9], the sequence $\mathsf{GP}(n)$ is not equal to EIS sequence 79566, and there appears to be no relation between gp-posets and C_4-free connected graphs.

We comment on some details in Table 1. The difference $\mathsf{P}(4) - \mathsf{SP}(4) = 1$ is witnessed by the N poset that is not series-parallel, whereas $\mathsf{P}(4) - \mathsf{IO}(4) = 1$ is witnessed by the $\mathsf{2{+}2}$ poset. The difference $\mathsf{P}(6) - \mathsf{GP}(6) = 5$ is witnessed by the structures in Proposition 31. The differences between $\mathsf{IP}(n)$ and $\mathsf{GPI}(n)$ for $n \leq 4$ have been discussed in Example 3.

[1] Our Julia code is available at `https://github.com/ulifahrenberg/pomsetproject/tree/main/code/20210618/`, and the data at `https://github.com/ulifahrenberg/pomsetproject/tree/main/data/`.

Table 1: Different types of posets and iposets on n points: all posets; sp-posets; interval orders; gp-posets; iposets; gp-iposets. The last line refers to the column sequence's index in Neil Sloane's Encyclopedia of Integer Sequences, `http://oeis.org/`, if available.

n	$\mathsf{P}(n)$	$\mathsf{SP}(n)$	$\mathsf{IO}(n)$	$\mathsf{GP}(n)$	$\mathsf{IP}(n)$	$\mathsf{GPI}(n)$
0	1	1	1	1	1	1
1	1	1	1	1	4	4
2	2	2	2	2	17	16
3	5	5	5	5	86	74
4	16	15	15	16	532	419
5	63	48	53	63	4068	2980
6	318	167	217	313	38.933	26.566
7	2045	602	1014	1903	474.822	289.279
8	16.999	2256	5335	13.943	7.558.620	3.726.311
9	183.231	8660	31.240	120.442		
10	2.567.284	33.958	201.608	1.206.459		
11	46.749.427	135.292	1.422.074			
EIS	112	3430	22493			

11. Conclusion

We have introduced posets with interfaces (iposets) with a new gluing operation that generalises the standard serial composition of posets, but identifies some of the maximal points of the first poset with some of the minimal points of the second. In the interpretation of posets as components of concurrent events, such interfaces allow events to continue across components or, alternatively, decompositions of posets with respect to synchronic cuts in time.

The idea of equipping posets or pomsets with interfaces in concurrency theory can be traced back to Winkowski [34], who introduces a gluing composition for pomsets without autoconcurrency where all maximal elements of the first pomset are merged with the corresponding minimal elements of the second. Our operation generalises this to interfaces consisting of subsets of maximal or minimal elements and gluing composition guided by interface identification rather than labels. Similar interface-based compositions of graphs and posets have been explored by Courcelle and Engelfriet [6], Fiore and Devesas Campos [11], and Mimram [24], but these do not introduce precedence when gluing and give rise to simpler algebraic structures, monoidal categories and PROPs, with strict interchange laws. Further, [11, 24] remove interfaces when gluing; whereas in our case, thinking of points as events in concurrent systems, we should rather keep them.

We have shown that iposets under gluing and parallel composition and with a suitable notion of subsumption form a 2-category with lax tensor. This generalises the concurrent monoids used in freeness results for series-parallel pomsets. Whether the iposets generated from singletons by finitary gluing and parallel compositions form the free 2-category with some adaptations remains open.

We have also shown that the hierarchy of gluing-parallel iposets defined by the alternation levels of the two compositions is infinite. This supports the idea that the algebra of these gp-iposets should be freely generated in one way or another. Further, our hierarchy lies above that for series-parallel posets and captures interval orders at its second alternation level.

Using a computer program, we have found five posets on 6 points, one on 8, and another five on 10 points, which are forbidden induced substructures for gp-posets. The five substructures on 6 points cannot be decomposed along interfaces, whereas the other six posets above are decomposable, but their components have their interfaces swapped. If such swapping were permitted for gp-iposets, then the six bigger forbidden substructures would disappear. Exploring the resulting *gluing-parallel-symmetric* iposets is left for future work.

We have recently defined languages of higher-dimensional automata as sets of ipomsets whose underlying posets are interval orders [10]. We conjecture that such automata are the machine model for ipomset languages, but leave this exploration to future work. A final avenue for future work is the development of a higher dimensional Kleene algebra for the languages for these automata and more general for languages of iposets that arise from lifting their 2-category to the powerset level.

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