Lecture Notes on Isomorphic Graphs

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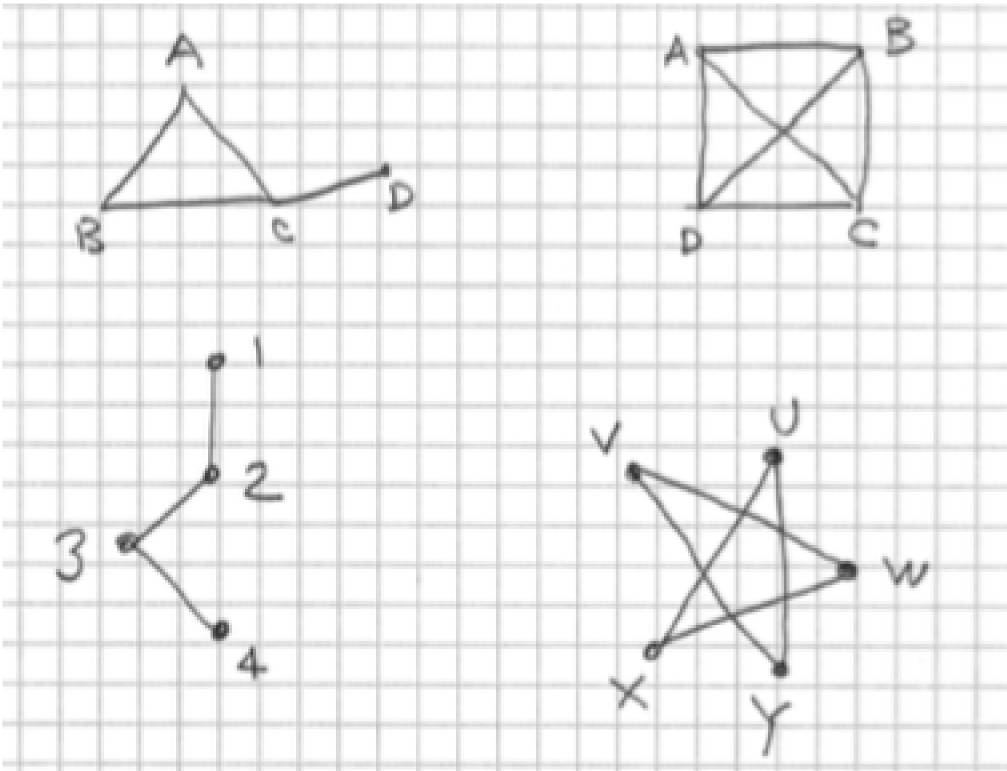
Graph theory is the study of graphs which model pairwise relations between objects.

Graph theory is a relatively new branch of mathematics. In 1736, Leonhard Euler published what is regarded as the first paper on Graph theory, *Seven Bridges of Königsberg*. The term *graph* was first introduced in the journal of Nature in 1878. The first Graph Theory textbook, (entitled *Graph Theory*) by Dénes König, appeared in 1936.

Graph Theory, when viewed as pure mathematics, has many as of yet open problems, as well as surprising applications in many areas: Electrical Engineering, Chemistry, Game theory, compiler technology, transportation, social networks, data bases, and many more.

1 Intuitive Examples



Intuitively, graphs are collections of vertices and connections between them.

Note that crossing lines are *not* connections.

2 Kinds of Graphs

Definition 2.1 (k-subsets)**.** If V is a set of n elements, ($|V| = n$), then the *k-subsets* are the subsets of V having exactly k elements. We denote the set of k-subsets as $\binom{V}{k}$.

$$\binom{V}{k} = \{u \subset V \mid |u| = k\} = \{u \in 2^V \mid |u| = k\}$$

Note that the size of $\binom{V}{k}$ is $\left|\binom{V}{k}\right| = \binom{|V|}{k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

We are particularly interested in $k = 2$, i.e., the $2 - subsets$, $\binom{V}{2}$.

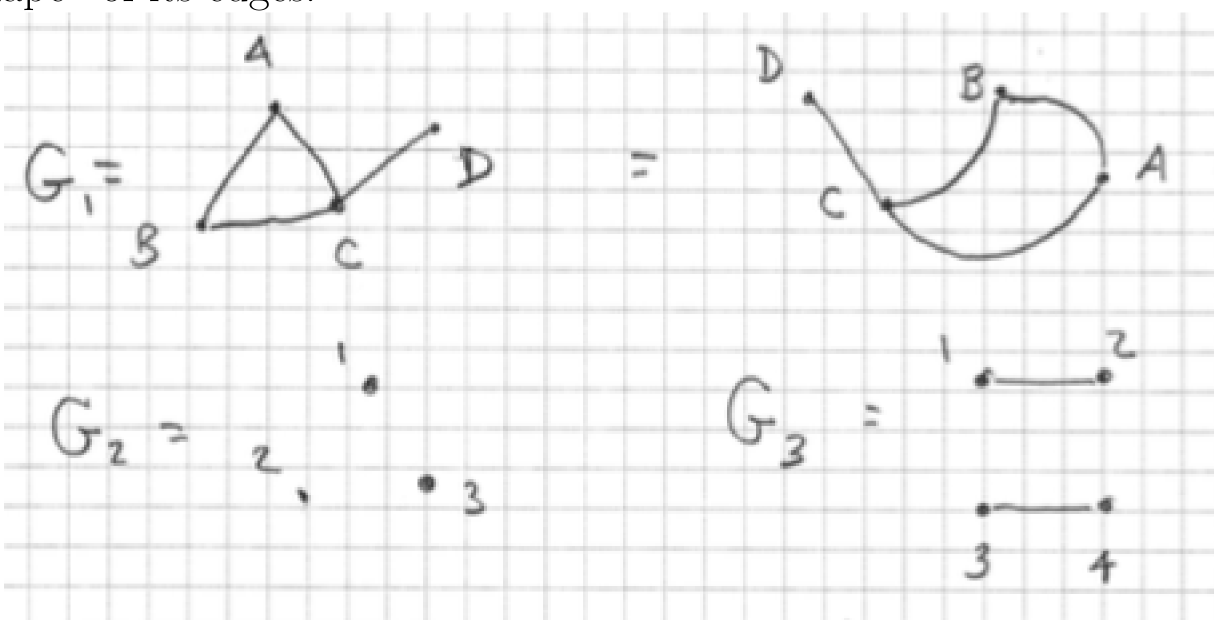
$$\left|\binom{V}{2}\right| = \frac{|V|!}{2!(|V|-2)!} = \frac{n \times (n-1)}{2}$$

Definition 2.2. A *simple graph*, (V, E), is an object consisting of two finite sets V and E, such that $V \neq \emptyset$, and $E \subset \binom{V}{2}$. V is called the *vertex set*, and each element of V is called a *vertex* or a *node*. E is called the *edge set*, and every element of E is called an *edge* or an *arc*.

Examples:

$$G_1 = \big(\{A, B, C, D\}, \{\{A, B\}, \{B, C\}, \{A, C\}, \{C, D\}\}\big)$$
$$G_2 = \big(\{1, 2, 3\}, \emptyset\big)$$

We can naturally represent graphs by drawing them. But a graph does not depend on the arbitrarily chosen "position" of its drawn vertices nor "shape" of its edges.



Some consequences of the definition of simple graph:

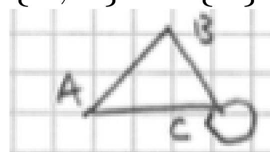
- A graph is not necessarily connected. E.g., G_3.

- E may be empty.

- V is never empty.

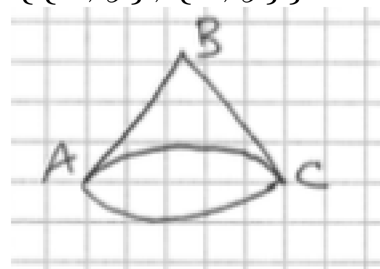
- Edges have no "direction", because $\{x, y\} = \{y, x\}$. We may also talk about *directed graphs*, but these are not simple graphs.

- There are no looping edges because $\{x, x\} = \{x\}$ is not of size 2.

  The following is not a simple graph: 

  .

- There are no parallel edges because $\{\{x, y\}, \{x, y\}\} = \{\{x, y\}\}$.

  The following is not a simple graph: 

  . Graphs which allow parallel edges or loops are called *multi-graphs*.

- A graph may have cycles.

  Definition 2.3. A *tree* is a connected acyclic simple graph.

  A collection of trees which share no vertex form a forest. More precisely, contrasted with a DAG (directed acyclic graph):

  Definition 2.4. A *forest* is an acyclic simple graph.

- Every edge has exactly two vertices—contrasted with a *hypergraph* whose edges may have one or more vertices.

The following terminology is a bit confusing.

Definition 2.5 (Adjacent vertices, and incident edges)**.**

- If $\{v_1, v_2\}$ is an edge of a graph, we say that $\{v_1, v_2\}$ *joins* or *connects* the vertices v_1 and v_2, and that vertices v_1 and v_2 are *adjacent* (to each other).

- The edge $\{v_1, v_2\}$ is *incident* to vertices v_1 and to v_2.

- Two edges incident to the same vertex are *incident* (to each other).

- A vertex incident to no edge is called *isolated*.

3 Graph Isomorphisms

A useless definition

Definition 3.1. *Equal graphs*: If $G = (V, E)$ and $G' = (V', E')$, then $G = G$' means that $V = V'$ and $E = E'$.

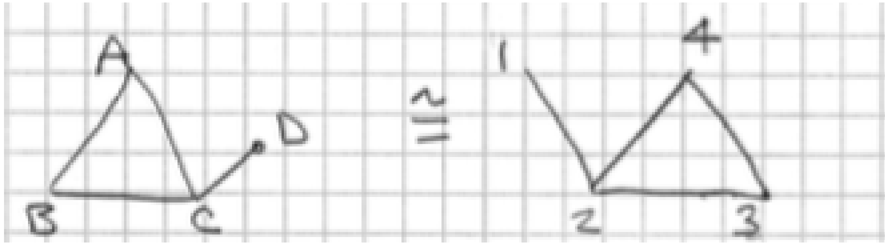
However, we rarely talk about equal graphs. Rather, we talk about isomorphic graphs.

Definition 3.2. If $G = (V, E)$ and $G' = (V', E')$, then G is said to be *isomorphic* to G', denoted $G \cong G'$, if there exists an edge preserving bijection, $h : V \to V'$ such that $\{x, y\} \in E$ if and only if $\{h(x), h(y)\} \in E'$.

Note that $\cong$ is an equivalence relation:

$$\begin{aligned}
&G \cong G, \\
&G \cong G' \text{ implies } G' \cong G, \text{ and} \\
&G \cong G', G' \cong G'' \text{ implies } G \cong G''
\end{aligned}$$

Note that h is not necessarily unique.



$$G_1 = (\{A, B, C, D\}, \{\{A, B\}, \{B, C\}, \{A, C\}, \{C, D\}\})$$
$$G_2 = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{4, 3\}, \{2, 4\}\})$$

Find an isomorphism:

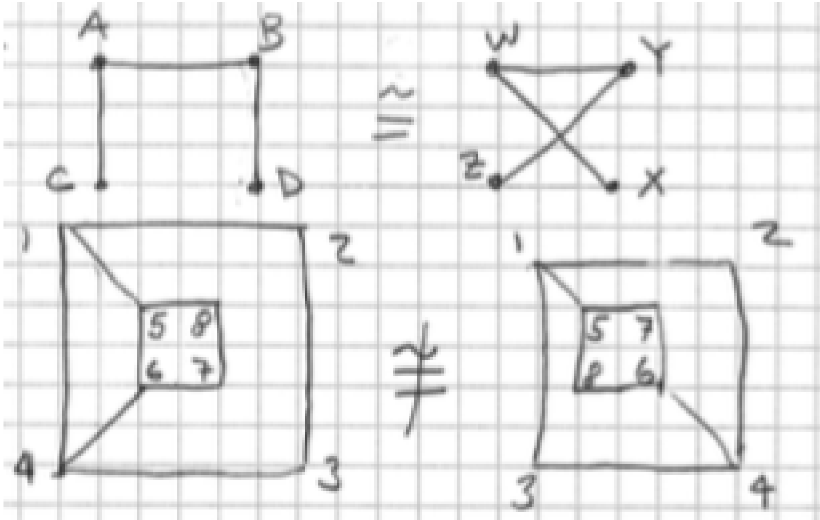
$$\begin{aligned} h(A) &= 3 \text{ (or 4)} \\ h(B) &= 4 \text{ (or 3)} \\ h(C) &= 2 \\ h(D) &= 1 \end{aligned}$$

In particular

$$\begin{aligned} \{h(A), h(B)\} &= \{3, 4\} \\ \{h(B), h(C)\} &= \{4, 2\} \\ \{h(A), h(C)\} &= \{3, 2\} \\ \{h(C), h(D)\} &= \{2, 1\} \end{aligned}$$

Deciding whether two given graphs are isomorphic is hard. Finding an isomorphism is difficult especially when you don't know whether one exists.

Call these G_6, G_7, G_8, and G_9



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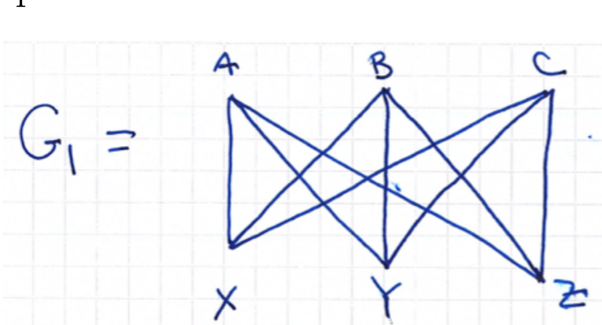
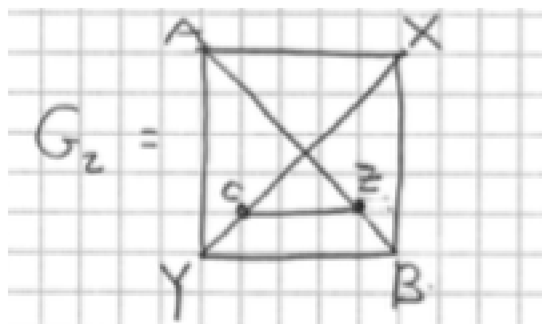
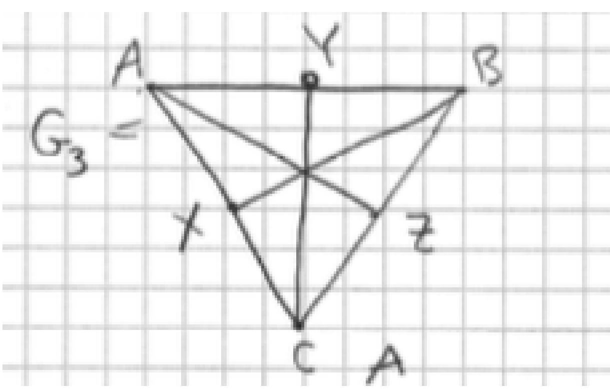
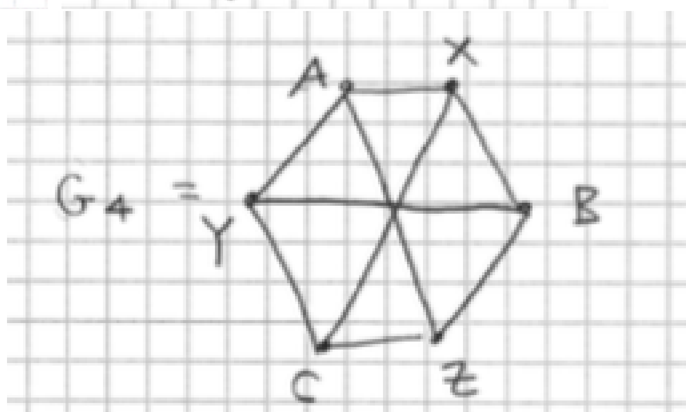
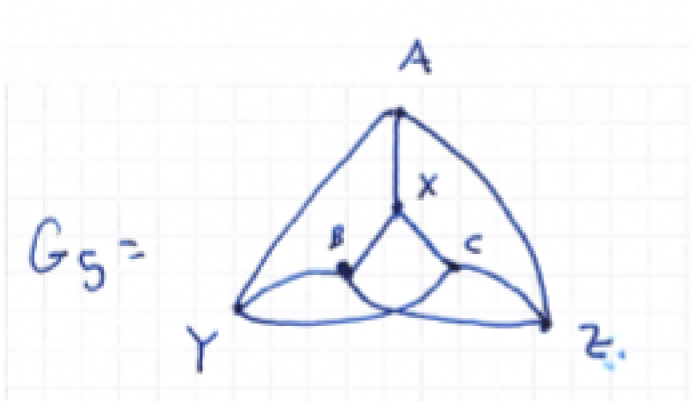
G_6, G_7, G_8, and G_9 are easy examples where we can intuitively decide whether the graphs are isomorphic. The following example is a bit more complicated.











$G_1 \cong G_2 \cong G_3 \cong G_4 \cong G_5$. This graph is called the *utility graph* and is denoted $K_{3,3}$ or UG.

4 Counting the graphs with n vertices

How many graphs exists with n vertices? If (V, E) is a graph, then, by definition, $E \subset \binom{V}{2}$. There are $2^{\binom{n}{2}}$ subsets of E. So there are potentially

$$2^{\binom{n}{2}} = 2^{\frac{n\times(n-1)}{2}} = \sqrt{2^{n\times(n-1)}}$$

graphs of size n, many of which are isomorphic.

The Online Encyclopedia of Integer Sequences lists this sequence as sequence-88.

$$\begin{aligned} A_{88} = \{&1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, \\ &165091172592, 50502031367952, 29054155657235488, \\ &31426485969804308768, 64001015704527557894928, \\ &245935864153532932683719776, \\ &1787577725145611700547878190848\} \end{aligned}$$

Some of these graphs are important enough to have names. We will look at N_n, K_n, $K_{n,m}$, and C_n in the following sections.

5 Determining whether graphs are isomorphic

Given two graphs with n vertices $G = (V, E), G' = (V', E')$ and a bijection $h : V \to V'$, it is straightforward to test whether h is an isomorphism. I.e,

1. test $\forall \{v_1, v_2\} \in E$ is $\{h(v_1), h(v_2)\} \in E'$, and
2. $\forall \{v_1, v_2\} \in E'$ is $\{h^{-1}(v_1), h^{-1}(v_2)\} \in E$?

Note that if $|E| = |E'|$, then $1 \implies 2$; because we are already supposing that h is a bijection.

However, there are $n!$ different bijections $h : V \to V'$; so finding the right h is difficult.

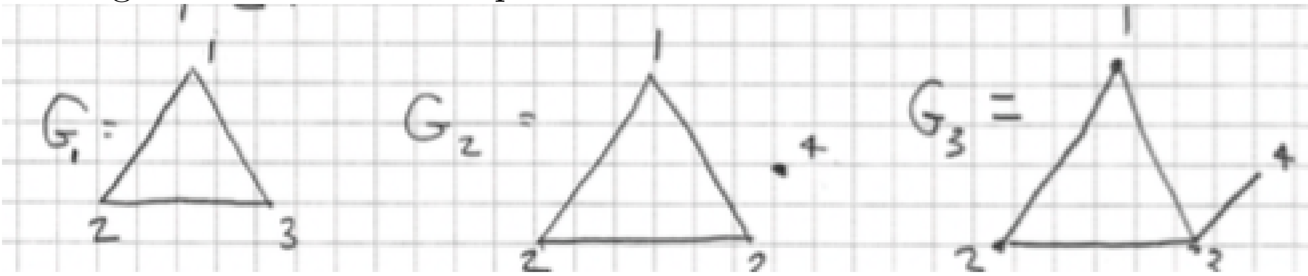
Definition 5.1. If two given graphs are not isomorphic, they are said to be *distinguishable* (from each other).

There are some tests for distinguishable graphs. But WARNING, they don't always work.

5.1 Test 1

Isomorphic graphs have the same number of vertices and edges.

So if two graphs differ in the number of vertices or edges, then they are distinguishable. For Example.



G_1 and G_2 differ in number of vertices. G_1 and G_2 differ in number of edges.

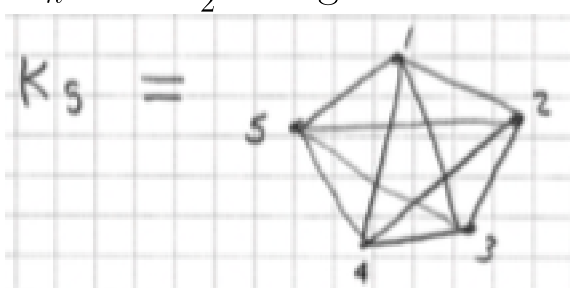
Definition 5.2 (The complete graph)**.** K_n is called the *complete* graph with n nodes. $K_n = (V, E)$ where E is the set of all two-element subsets of V.

Another way to define K_n is

$$K_n = (V, \binom{V}{2})$$

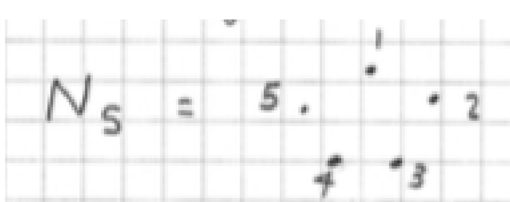
Note that we say *the complete graph*, because we consider all isomorphic graphs to be the same. It does not normally make sense to talk about two different complete graphs with the same number of vertices, even if the vertices are labeled differently, or if they are drawn differently on the page.

Note that K_n has $\frac{n(n-1)}{2}$ edges.

Example: 

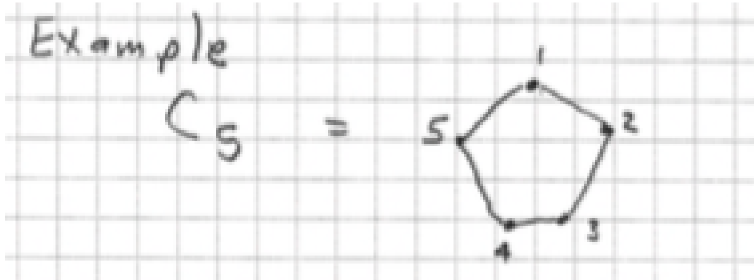
Definition 5.3. N_n is called the *null graph* and is the graph with n vertices and no edges.

$$N_n = (V, \emptyset)$$

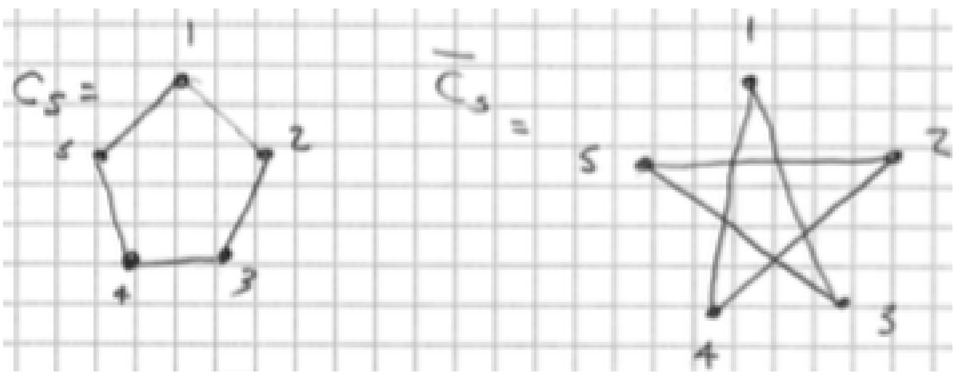
Example: 

Definition 5.4. C_n is called the *cycle graph* with n vertices.

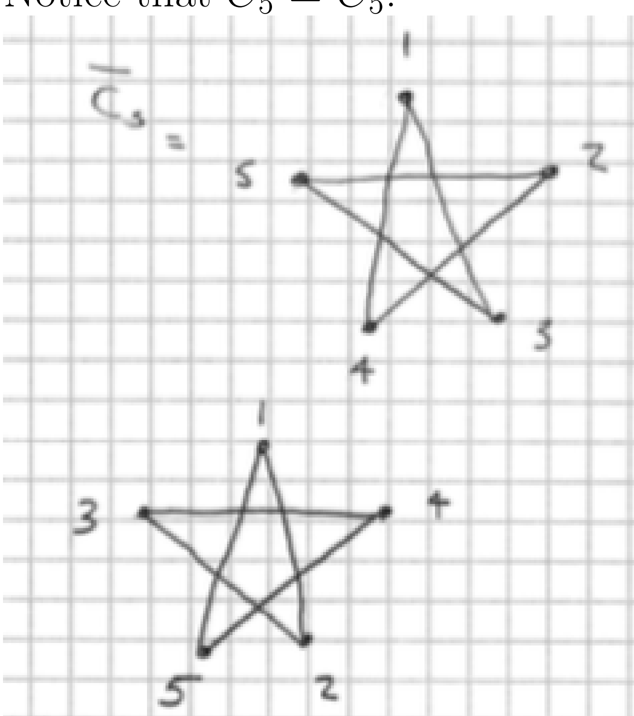
$$C_n = \big(\{v_1, v_2, ...v_n\}, \{\{v_1, v_2\}, \{v_2, v_3\}, ...\{v_{n-1}, v_n\}, \{v_n, v_1\}\}\big)$$

Example: 

Definition 5.5. If $G = (V, E)$ is a graph, the *complement graph* of G, denoted $\overline{G}$, is the graph (V, E') where E' is the set of all two-element subsets of V which are not in E, and no others.



Notice that $C_5 \cong \overline{C_5}$.



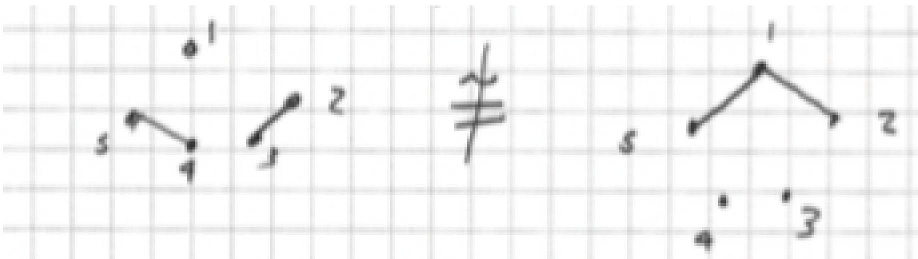
Another example: $N_5 \cong \overline{K_5}$ and $\overline{N_5} \cong K_5$.

5.2 Test 2

Since $G \cong G'$ if and only if $\overline{G} \cong \overline{G'}$ we can test the smaller one.

5.3 Test 3

Isomorphic graphs have the same number of "pieces". We won't define pieces.



5.4 Test 4

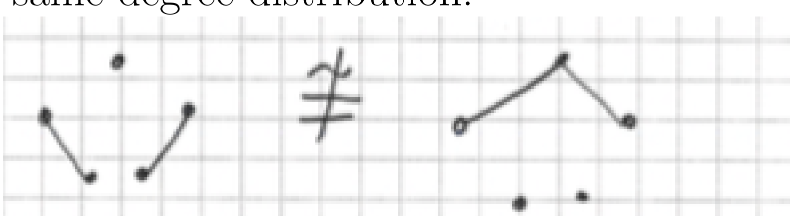
Definition 5.6. The *degree* of a vertex is the number of neighbors (number of incident edges).

Note that the sum of the degrees of the vertices is equal to twice $|E|$.

Isomorphic graphs have the same vertex degree distribution.

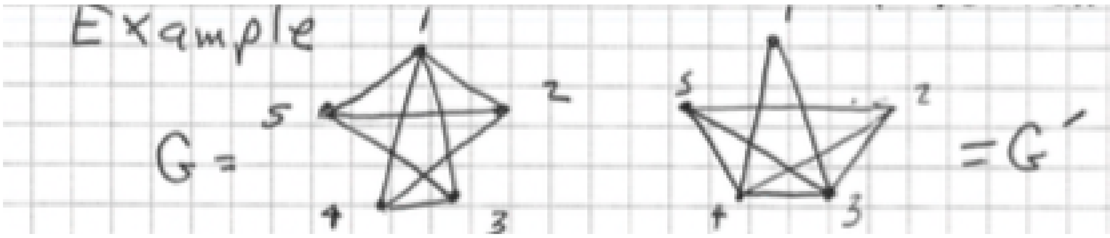
I.e., if $G \cong G'$ and G as m vertices with degree d, then so does G'.

Exercise for the student: Find two distinguishable graphs which share the same degree distribution.

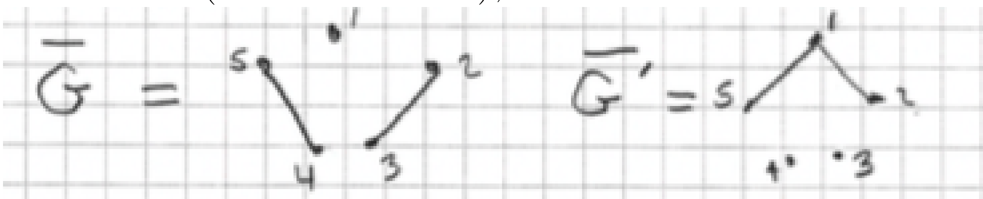


1 vertex of degree 0	2 vertices of degree 0
4 vertices of degree 1	2 vertices of degree 1
	1 vertex of degree 2

Another example. is $G \cong G'$?



Use test 2 (find $\overline{G}$ and $\overline{G'}$), then either test 3 or test 4.



5.5 Test 5

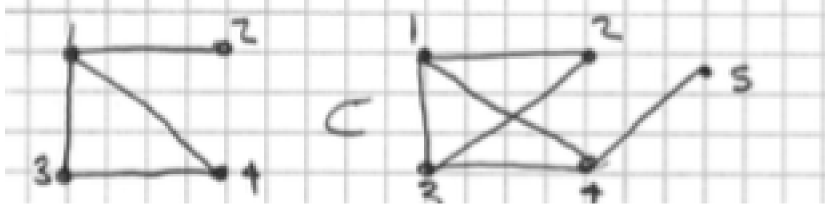
Definition 5.7. If $G = (V, E)$ and $G' = (V', E')$ are graphs such that $V' \subset V$ and $E' \subset E$, then G' is called a *subgraph* of G, denoted $G' \subset G$.

Recall that a graph is not a set, so this is really new (abuse of) notation $G' \subset G$.

Warning, if $G' = (V', E')$ with $V' \subset V$ and $E' \subset E$, then G' is not necessarily a subgraph of G. Example

$$G = \big(\{1,2,3\}, \{\{1,2\},\{2,3\}\}\big)$$
$$G' = \big(\{1,2\}, \{\{2,3\}\}\big)$$

G' is not a subgraph of G, because G' is not a graph.

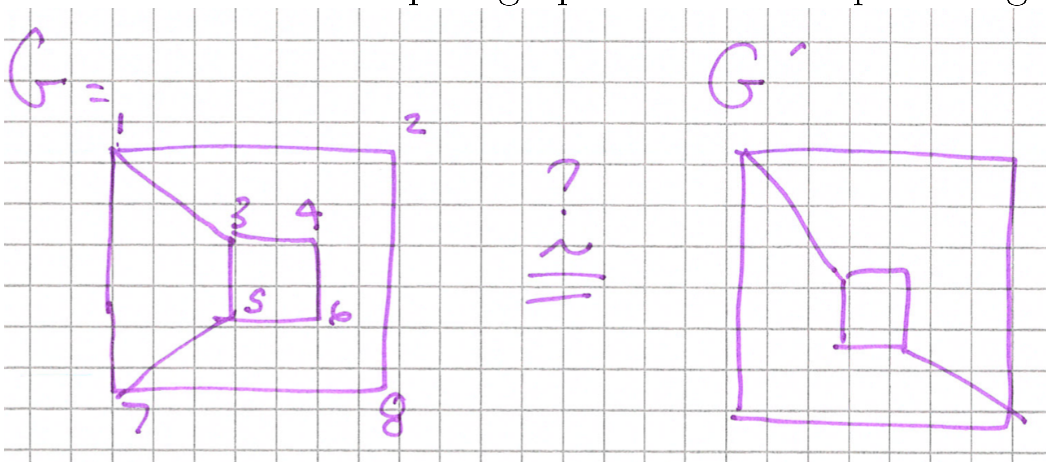
Example: 

Another example: N_n is a subgraph of every n-vertex graph.

Another example: Every n-vertex graph is a subgraph of K_n. So $N_n \subset C_n \subset K_n$.

If $G \cong G'$ and $S \subset G$, then there exists $S' \subset G'$ such that $S \cong S'$.

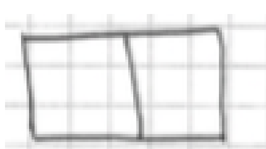
In other words isomorphic graphs have isomorphic subgraphs.



G	G'
10 edges	10 edges
8 vertices	8 vertices
4 vertices of degree 2	4 vertices of degree 2
4 vertices of degree 3	4 vertices of degree 3

We would like to argue that $G \ncong G'$. If we look at tests 1 through 4, we see that none of them helps.

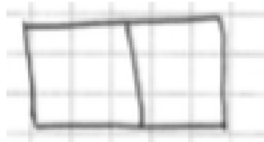
5.5.1 Argument 1

Take $S =$ , which has 7 edges and 6 vertices. Notice that $S \subset G$. We would like to show that $S \not\subset G'$.

Can we find $S' \subset G'$ with 7 edges and 6 vertices? I.e., can we remove 2 vertices of G' including their incident edges, and be left with exactly 7 edges? The answer is no, and here's why.

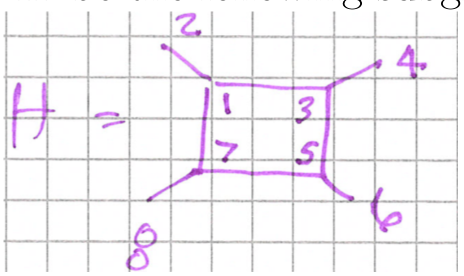
G only has vertices of degrees 2 and 3. If we remove any vertex of degree 3, then we remove 3 edges (leaving 7 edges), and removing any addition vertex would remove at least one edge, leaving at most 6.

Therefore, the only hope is to remove two vertices of degree 2. G' has no adjacent vertices of degree 2, so removing 2 vertices of degree 2 would remove 4 edges, again leaving 6 edges maximum.

This means there is no  subgraph of G'. Therefore $G \ncong G'$

5.5.2 Argument 2

Let H be the following subgraph of G.



This graph has the degree distribution:

8 vertices
8 edges
4 vertices with degree=3.
4 vertices with degree=1.

We will argue that G' has no such subgraph.

G' has 8 vertices and 10 edges. So we must remove 2 edges, and no vertices. However, G' has 4 vertices of degree 3. So we must remove edges not incident to a 3-degree edge. This is impossible because all edges of G' are adjacent to a 3-degree vertex. Therefore, $H \not\subset G'$, and thus $G \ncong G'$.

6 Walks and Cycles

Definition 6.1 (walks and cycles)**.** A *walk*, $v_1, v_2, ..., v_n$, in a graph is a sequence of vertices (not necessarily exhaustive nor distinct) for which $\{v_i, v_{i+1}\}$ is an edge for every $1 \leq i < n$.

The walk is said to *join* v_1 and v_n.

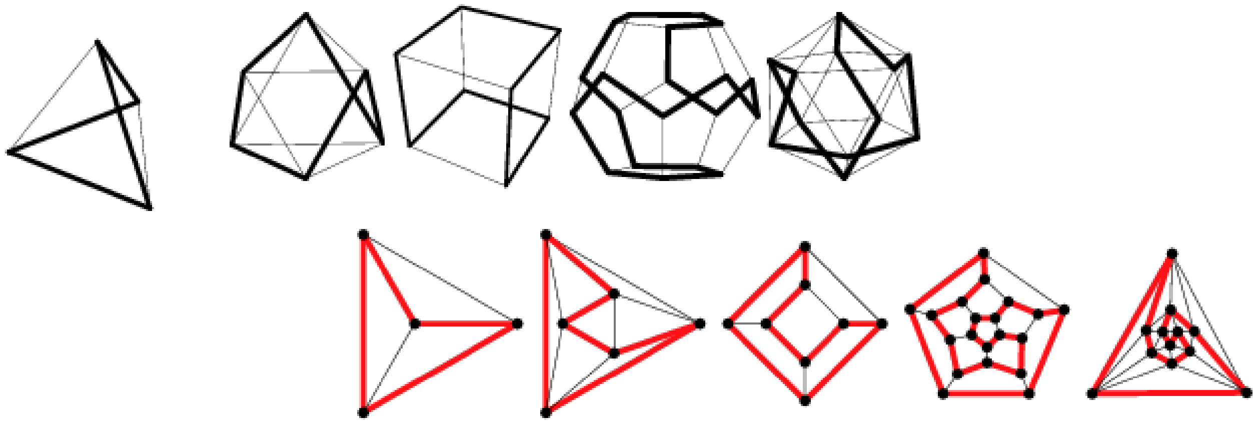
If $v_1 = v_n$, the walk is called a *cycle* or a *closed walk*. Otherwise the walks is said to be *open*.

6.1 Hamiltonian Walks

Definition 6.2 (Hamiltonian Walk)**.** A *Hamiltonian Walk* is a walk which visits every vertex of a graph exactly once.

Definition 6.3 (Hamiltonian Cycle)**.** A *Hamiltonian Cycle*, is a cycle, $\{v_1, v_2, ..., v_n\}$ for which $v_1, v_2, ..., v_{n-1}$ is a Hamiltonian walk.

The following image taken from Wolfram MathWorld illustrates some Hamiltonian cycles.



The problem of finding a Hamiltonian cycle is NP-complete. The way to find one is to exhaust the possibilities. The famous *traveling salesman problem* is to find a shortest Hamiltonian cycle on a weighted graph.

6.1.1 Visiting all permutations

In Computer Programming we sometimes need to generate or at least visit all permutations. This is of course computationally expensive. For small number of objects, this can be done using a recursive function. There are other ways. Many are explained in Donald Knuth: *Art of Computer Programming. Volume 4* Section 7.2.1.2. Also available as Volume 4 Fascicle 2.

One thing to keep in mind is sometimes you will be asked to generate all permutations. This is usually not possible, so whoever is asking you to do it needs to rethink his request. Easier is to *visit* each permutation. Or *visit some* permutations, either a limited number, or until some time-out.

6.1.2 Counting superpermutations

See the related Numberphile video on YouTube.
`https://www.youtube.com/watch?v=wJGE4aEWc28`.

Definition 6.4 (superpermutation)**.** A *superpermutation* of n symbols is a string which contains each permutation of the n symbols as a substring.

For example for the set $\{1, 2\}$, `1221` and `121` are both superpermutations, but `121` is shorter. The string `1231221321` is a superpermutation of the set $\{1, 2, 3\}$.

Quotation from Wikipedia superpermutations: On 20 October 2018, by adapting a construction by Aaron Williams for constructing Hamiltonian paths through the Cayley graph of the symmetric group, Greg Egan devised an algorithm to produce superpermutations of length

$$n! + (n-1)! + (n-2)! + (n-3)! + n - 3\,.$$

Up to 2018, these were the smallest superpermutations known for $n \geq 7$. However, on 1 February 2019, Bogdan Coanda announced that he had found a superpermutation for $n = 7$ of length 5907, or

$$\big(n! + (n-1)! + (n-2)! + (n-3)! + n - 3\big) - 1\,,$$

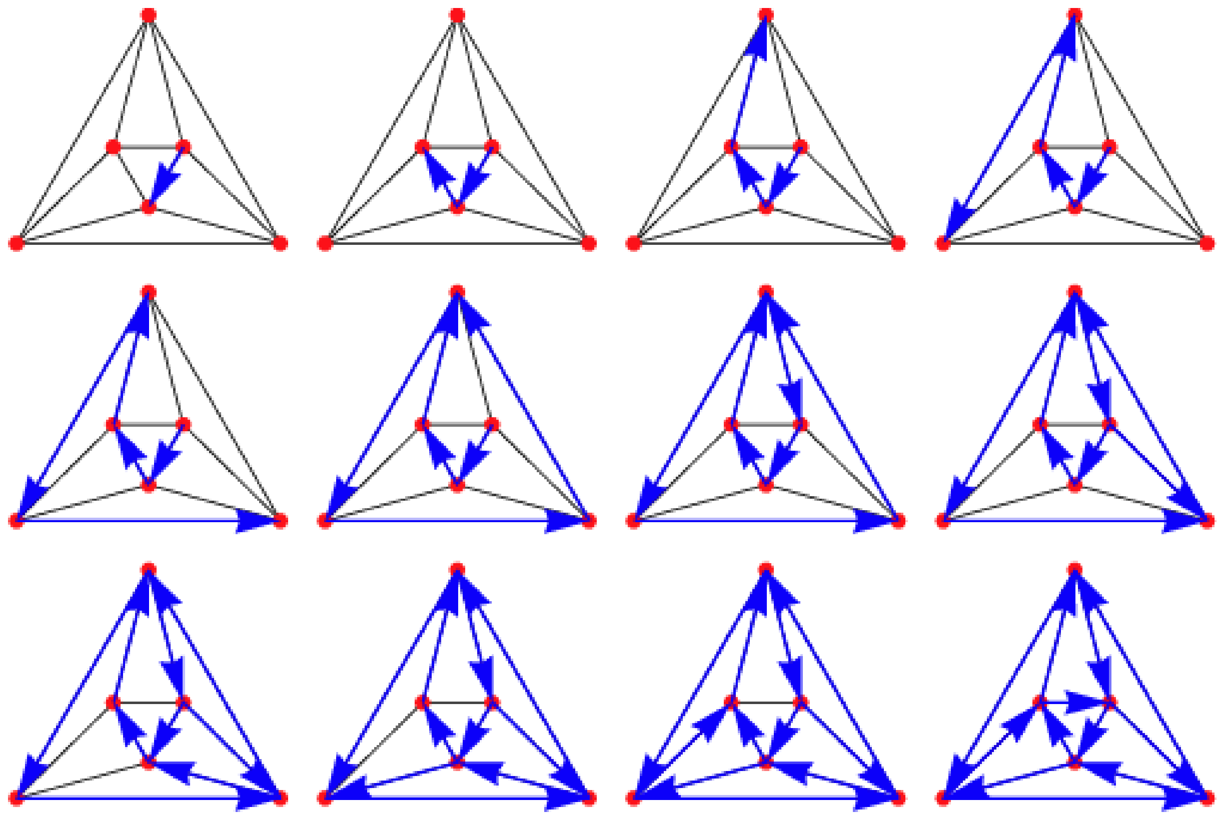
which was a new record. On 27 February 2019, using ideas developed by Robin Houston, Egan produced a superpermutation for $n = 7$ of length 5906. Whether similar shorter superpermutations also exist for values of $n > 7$ remains an open question.
`https://en.wikipedia.org/wiki/Superpermutation`

6.2 Eulerian Walks

Definition 6.5 (Eulerian Walk)**.** An *Eulerian walk* is a walk which visits every edge exactly once.

Definition 6.6 (Eulerian Cycle)**.** An *Eulerian cycle* is a cycle which visits every edge exactly once.

The following image taken from Wolfram MathWorld illustrates some Eulerian cycles.

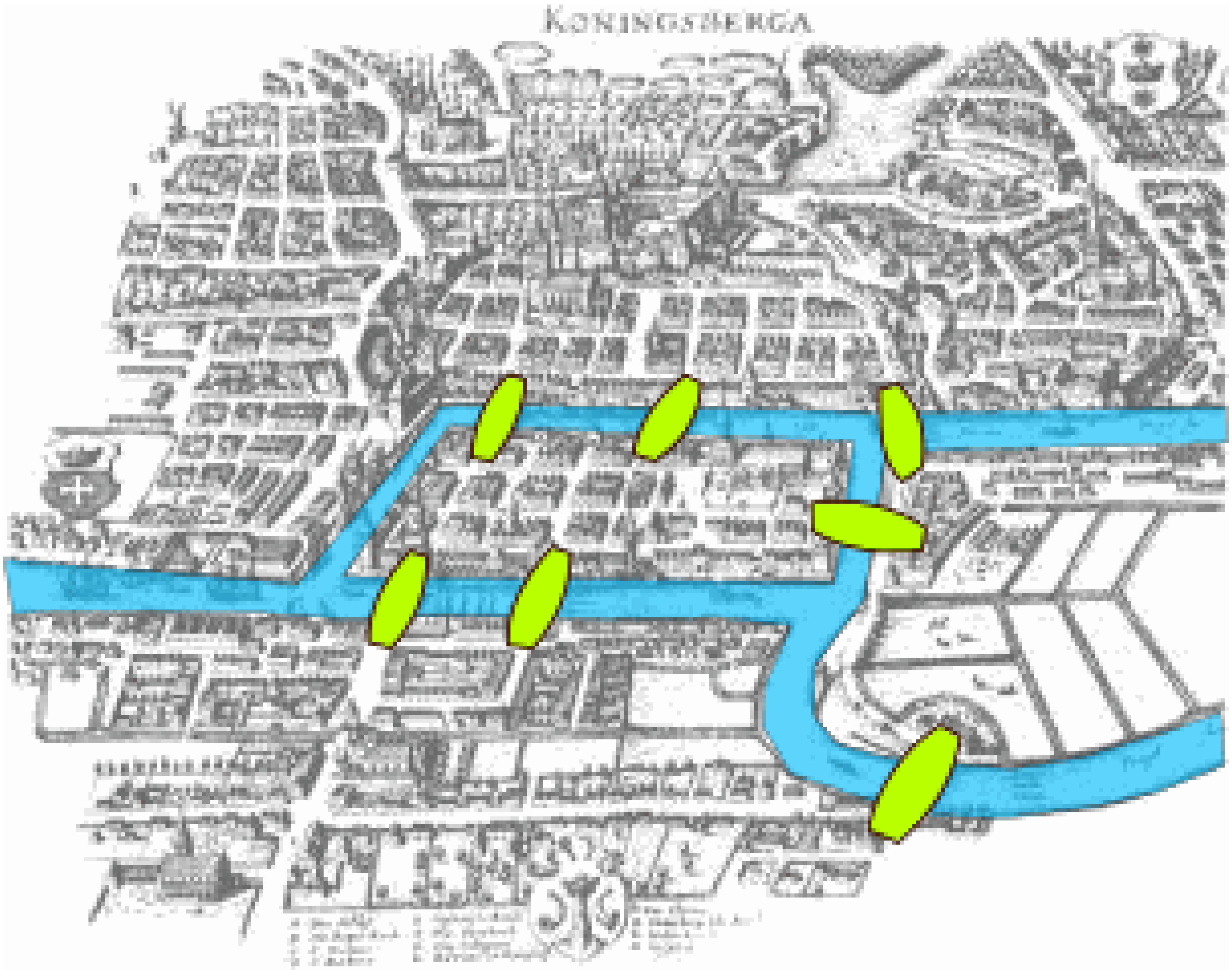


However, finding an Eulerian cycle is straightforward.

Any graph which has only even-degree vertices, contains an Eulerian cycle.

If a graph has exactly two odd-degree vertices, then it has an Eulerian walk which connects the two odd-degree vertices.

A famous problem in graph theory, *Seven Bridges of Königsberg* is the question of finding an Eulerian cycle which crosses each of the Königsberg exactly once.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