Graph Isomorphism

September 22, 2026

Start of video 1:

Definitions

Graph Theory

Definition (graph theory)

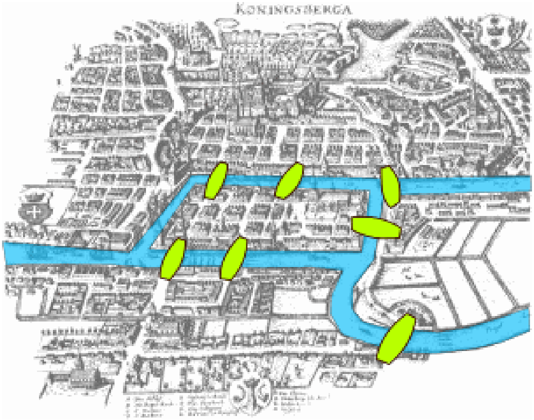
Graph theory is the study of graphs which model pairwise relations between objects.

Graph theory is a relatively new branch of mathematics. In 1736, Leonhard Euler published what is regarded as the first paper on Graph theory, *Seven Bridges of Königsberg*.

Graph Theory

The term *graph* was first introduced in the journal of Nature in 1878. The first Graph Theory textbook, (entitled *Graph Theory*) by Dénes König, appeared in 1936.

Graph Theory



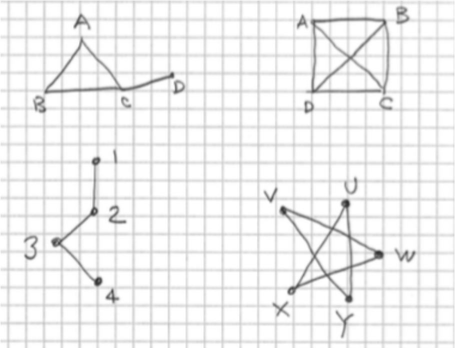
Graph Theory

Graph Theory, when viewed as pure mathematics, has many as of yet open problems.

Graph Theory

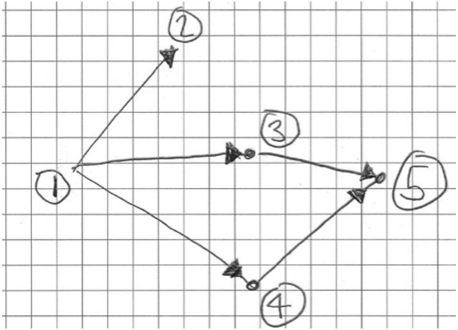
Surprising applications in many areas: Electrical Engineering, Chemistry, Game theory, compiler technology, transportation, social networks, data bases, and many more.

Intuitive Examples



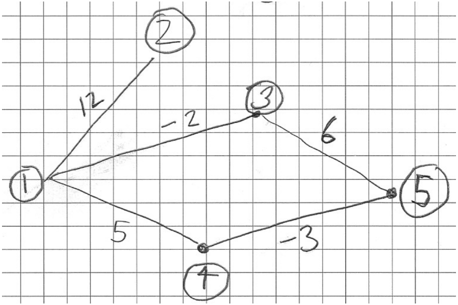
Intuitively, graphs are collections of vertices and connections between them.
Note that crossing lines are *not* connections.

Other genre of graphs



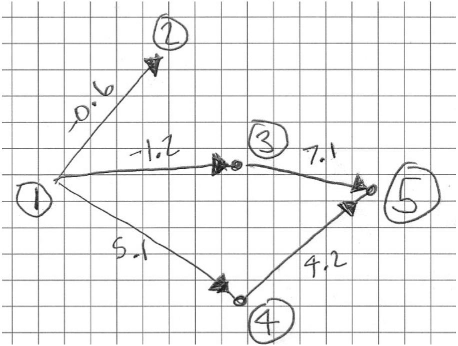
Edges may be directed. This is called a *directed graph*.

Other genre of graphs



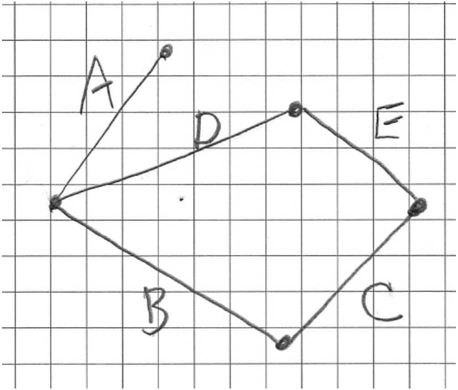
Edges may have weights assoicated with them. This is called a *weighted graph*.

Other genre of graphs



Edges may simultaneously have weights and directions. This is not surprisingly called a *weighted directed graph*.

Other genre of graphs



Sometimes it is useful to refer to edges by name, rather than by endpoint. This is just notation. Do not confused this with a weighted graph.

k-subsets

Definition (k-subsets)

If V is a set of n elements, ($|V| = n$), then the *k-subsets* are the subsets of V having exactly k elements. We denote the set of k-subsets as $\binom{V}{k}$.

$$\binom{V}{k} = \{u \subset V \mid \quad |u| = k\} = \{u \in 2^V \mid |u| = k\}$$

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$$\left|\binom{V}{3}\right| = \binom{|V|}{3} = \binom{n}{3} = \frac{n!}{3!(n-3)!}$$
$$\left|\binom{V}{4}\right| = \binom{|V|}{4} = \binom{n}{4} = \frac{n!}{4!(n-4)!}$$

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We are interested in $k = 2$, i.e., the $2-$ *subsets*, $\binom{V}{2}$.

$$\left|\binom{V}{2}\right| = \frac{n!}{2!(n-2)!} = \frac{n \times (n-1)}{2}$$

Simple graph

Definition

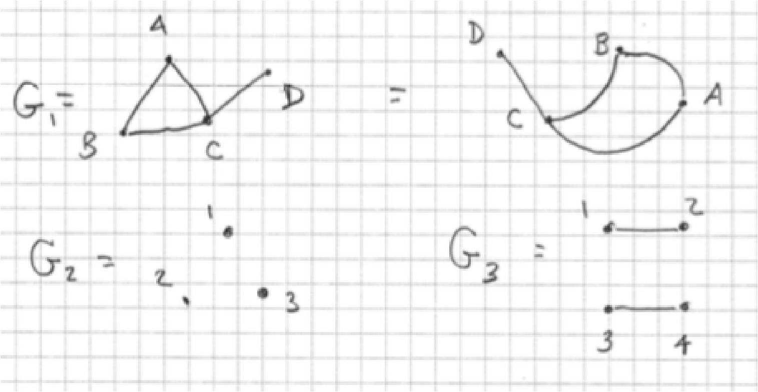
A *simple graph*, (V, E), is an object consisting of two finite sets V and E, such that $V \neq \emptyset$, and $E \subset \binom{V}{2}$. V is called the *vertex set*, and each element of V is called a *vertex* or a *node*. E is called the *edge set*, and every element of E is called an *edge* or an *arc*.

Examples:

$$G_1 = \big(\{A, B, C, D\}, \{\{A, B\}, \{B, C\}, \{A, C\}, \{C, D\}\}\big)$$
$$G_2 = \big(\{1, 2, 3\}, \emptyset\big)$$

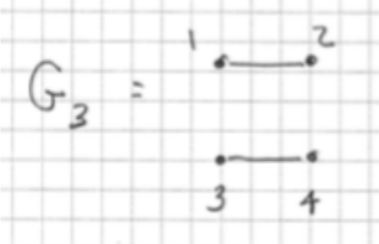
Drawing a graph

We can naturally represent graphs by drawing them. But a graph does not depend on the arbitrarily chosen "position" of its drawn vertices nor "shape" of its edges.



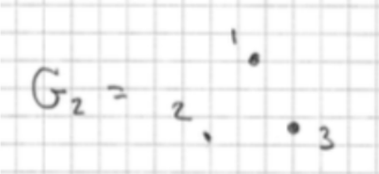
Consequences of definition

A graph is not necessarily connected.



Consequences of definition

E may be empty.

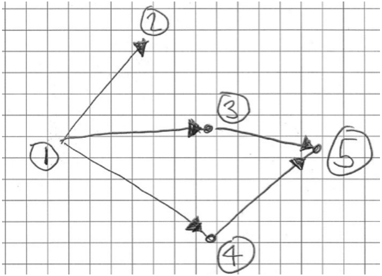


Consequences of definition

V is never empty in our treatment.
However, read carefully the definition of the author. Some authors may allow $V = \emptyset$.

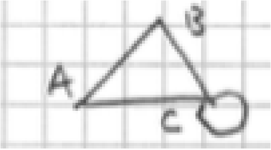
Consequences of definition

Edges have no "direction", because $\{x, y\} = \{y, x\}$. We may also talk about *directed graphs*, but these are not simple graphs.



Consequences of definition

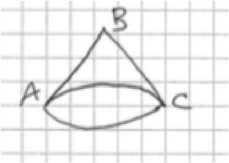
There are no self-looping edges because $\{x, x\} = \{x\}$ is not of size 2. The following is not a simple graph:



.

Consequences of definition

There are no parallel edges because $\{\{x, y\}, \{x, y\}\} = \{\{x, y\}\}$.

The following is not a simple graph: .

Consequences of definition

Every edge has exactly two vertices—contrasted with a *hypergraph* whose edges may have one or more vertices.

Neighbors

If $\{v_1, v_2\}$ is an edge of a graph, we say that $\{v_1, v_2\}$ *connects* the vertices v_1 and v_2, and that vertices v_1 and v_2 are *neighbors*.

Neighbors

A vertex with no neighbor is called *isolated*.

End of video

Start of video 2:

Isomorphism

Equal Graphs

A useless definition

Definition

Equal graphs: If $G = (V, E)$ and $G' = (V', E')$, then $G = G$' means that $V = V'$ and $E = E'$.

However, we rarely talk about equal graphs. Rather, we talk about isomorphic graphs.

Isomorphic graphs

Definition

If $G = (V, E)$ and $G' = (V', E')$, then G is said to be *isomorphic* to G', denoted $G \cong G'$, if there exists an edge preserving bijection, $h : V \to V'$ such that $\{x, y\} \in E$ if and only if $\{h(x), h(y)\} \in E'$.

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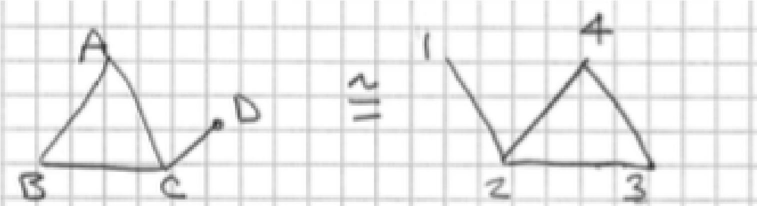
Note that $\cong$ is an equivalence relation:

$$\begin{aligned} &G \cong G, \\ &G \cong G' \text{ implies } G' \cong G, \text{ and} \\ &G \cong G', G' \cong G'' \text{ implies } G \cong G'' \end{aligned}$$

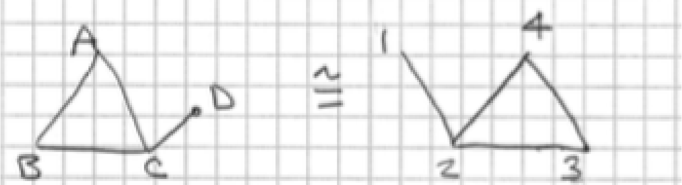
Isomorphic graphs

Note that the bijection, h, is not necessarily unique.



$$G_1 = (\{A, B, C, D\}, \{\{A, B\}, \{B, C\}, \{A, C\}, \{C, D\}\})$$
$$G_2 = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{4, 3\}, \{2, 4\}\})$$

Find an isomorphism



$$h(A) = 3 \text{ (or 4)}$$
$$h(B) = 4 \text{ (or 3)}$$
$$h(C) = 2$$
$$h(D) = 1$$

In particular

$$\{h(A), h(B)\} = \{3, 4\}$$
$$\{h(B), h(C)\} = \{4, 2\}$$
$$\{h(A), h(C)\} = \{3, 2\}$$
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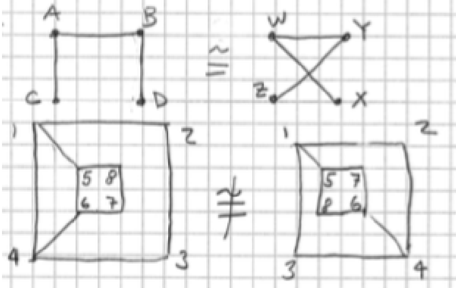
Finding an isomorphism

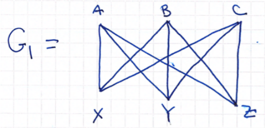
Deciding whether two given graphs are isomorphic is hard. Finding an isomorphism is difficult especially when you don't know whether one exists.

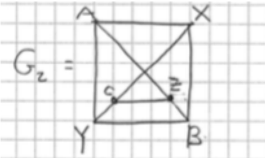


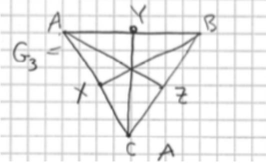
These are easy examples where we can intuitively decide whether the graphs are isomorphic.

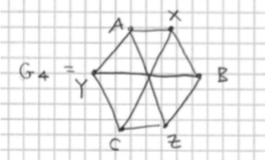
Finding an isomorphism

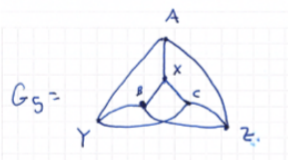
Which of these are isomorphic?











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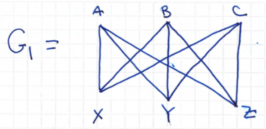
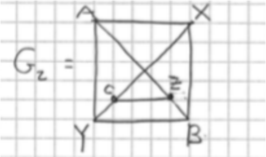
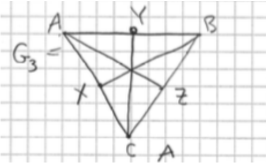
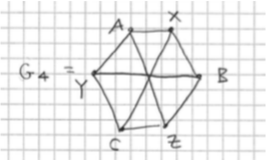
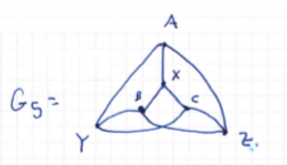






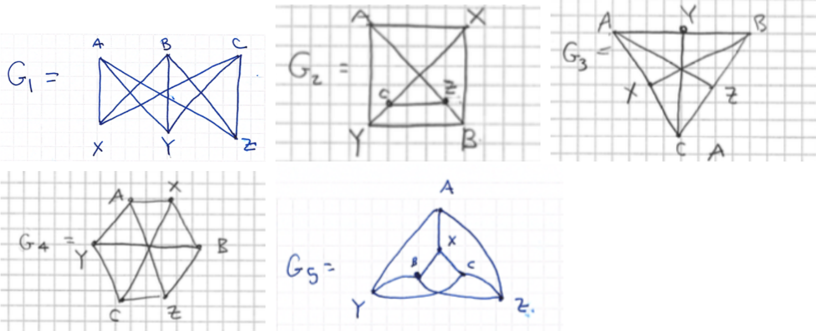




$G_1 \cong G_2 \cong G_3 \cong G_4 \cong G_5$.

Finding an isomorphism

Which of these are isomorphic?



This graph, denoted $K_{3,3}$ or UG, is called the *utility graph*.

Counting graphs

How many graphs exist with n vertices? If (V, E) is a graph, then, by definition, $E \subset \binom{V}{2}$. There are $2^{\binom{n}{2}}$ subsets of E. So there are potentially

$$2^{\binom{n}{2}} = 2^{\frac{n\times(n-1)}{2}} = \sqrt{2^{n\times(n-1)}}$$

graphs of size n, many of which are isomorphic.

Counting graphs

The Online Encyclopedia of Integer Sequences lists this sequence as sequence-88.

$$A_{88} = \{1, 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12005168, 1018997864, 165091172592, 50502031367952, 29054155657235488, 31426485969804308768, 64001015704527557894928, 245935864153532932683719776, 1787577725145611700547878190848\}$$

Counting graphs

Some of these graphs are important enough to have names.
We will look at N_n, K_n, $K_{n,m}$, and C_n in the following sections.

End of video

Start of video 3:

Connected graphs

Walks and Cycles

Definition (walks and cycles)

A *walk*, $v_1, v_2, ..., v_n$, in a graph is a sequence of vertices (not necessarily exhaustive nor distinct) for which $\{v_i, v_{i+1}\}$ is an edge for every $1 \leq i < n$.

The walk is said to *join* v_1 and v_n.

If $v_1 = v_n$, the walk is called a *cycle* or a *closed walk*. Otherwise the walks is said to be *open*.

Connectedness

Definition

A graph is said to be *connected* or (*vertex-connected*) if there is a walk between any two given vertices.

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A graph is said to be *edge-connected* if given two edges, there exists a walk which traverses both edges.

Connectedness

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Definition

A graph is said to be *edge-connected* if given two edges, there exists a walk which traverses both edges.

The two definitions conincide if there are no isolated vertices, i.e., no vertex with no neighbors.

Connectedness

Give an example of a graph which is connected but not edge-connected

Connectedness

Give an example of a connected graph with an isolated vertex.

End of video

Start of video 4:

Standard graphs

Standard graphs and isomorphism

Definition (The complete graph)

K_n is called the *complete* graph with n vertices. $K_n = (V, E)$ where E is the set of all two-element subsets of V.
Another way to define K_n is

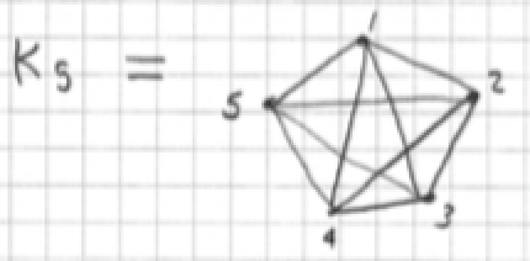
$$K_n = \left(V, \binom{V}{2} \right)$$

Standard graphs and isomorphism

Note that we say *the complete graph*, because we consider all isomorphic graphs to be the same. It does not normally make sense to talk about two different complete graphs with the same number of vertices, even if the vertices are labeled differently, or if they are drawn differently on the page.

Standard graphs and isomorphism

Note that K_n has $\frac{n(n-1)}{2}$ edges.

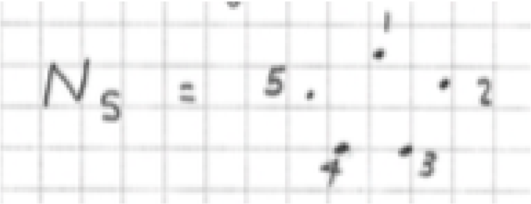
Example: 

Standard graphs and isomorphism

Definition

N_n is called the *null graph* and is the graph with n vertices and no edges.

$$N_n = (V, \emptyset)$$

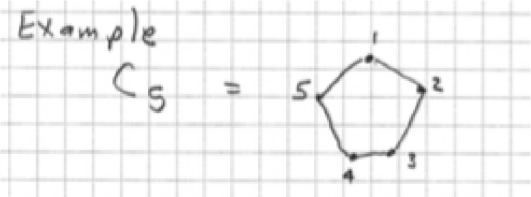
Example: 

Standard graphs and isomorphism

Definition

C_n is called the *cycle graph* with n vertices.

$$C_n = \left(\{v_1, v_2, \ldots v_n\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{n-1}, v_n\}, \{v_n, v_1\}\}\right)$$

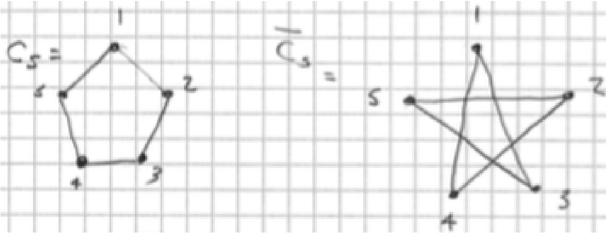
Example: 

Standard graphs and isomorphism

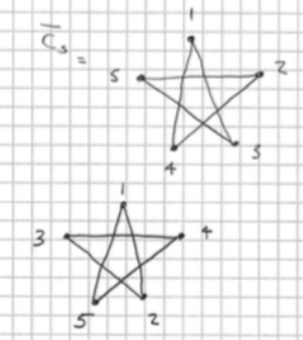
Definition

If $G = (V, E)$ is a graph, the *complement graph* of G, denoted $\overline{G}$, is the graph (V, E') where E' is the set of all two-element subsets of V which are not in E, and no others.

$$\overline{G} = \left(V, \binom{V}{2} \setminus E\right)$$



Standard graphs and isomorphism



Notice that $C_5 \cong \overline{C_5}$.

Another example: $N_5 \cong \overline{K_5}$ and $\overline{N_5} \cong K_5$.

End of video

Start of video 5:

Disproving Isomorphisms

Testing for isomorphism

Given two graphs with n vertices $G = (V, E), G' = (V', E')$ and a bijection $h : V \to V'$, it is straightforward to test whether h is an isomorphism. I.e,

1. test $\forall\{v_1, v_2\} \in E$ is $\{h(v_1), h(v_2)\} \in E'$, and
2. $\forall\{v_1, v_2\} \in E'$ is $\{h^{-1}(v_1), h^{-1}(v_2)\} \in E$?

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Does $1 \implies 2$?

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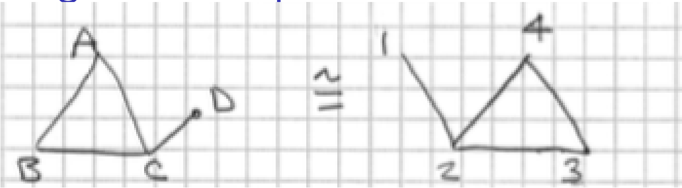
2. $\forall\{v_1, v_2\} \in E'$ is $\{h^{-1}(v_1), h^{-1}(v_2)\} \in E$?

Does $1 \implies 2$? No. Consider the counterexample:: $E' = \emptyset$. However;

if $|E| = |E'|$, then $1 \implies 2$; because we are already supposing that h is a bijection.

Testing for isomorphism



$$h(A) = 3 \text{ (or 4)}$$
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$$h(C) = 2$$
$$h(D) = 1$$

In particular

$$\{h(A), h(B)\} = \{3, 4\}$$
$$\{h(B), h(C)\} = \{4, 2\}$$
$$\{h(A), h(C)\} = \{3, 2\}$$
$$\{h(C), h(D)\} = \{2, 1\}$$

Testing for isomorphism

However, there are $n!$ different bijections $h : V \to V'$.

Finding the right h is difficult.

Distinguishable

Definition

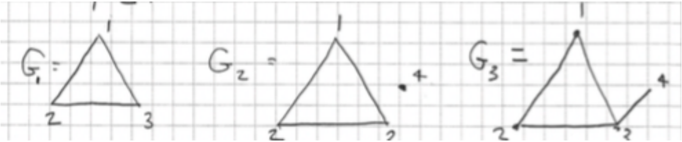
If two given graphs are not isomorphic, they are said to be *distinguishable* (from each other).

There are some tests for distinguishable graphs. But WARNING, they don't always work.

Test 1

Test 1: Isomorphic graphs have the same number of vertices and edges.

If two graphs differ in the number of vertices or edges, then they are distinguishable. For Example.



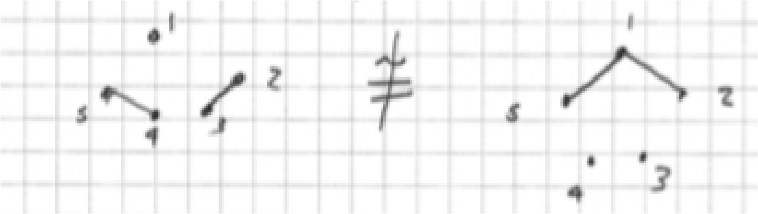
G_1 and G_2 differ in number of vertices. G_1 and G_2 differ in number of edges.

Test 2

Test 2: Since $G \cong G'$ if and only if $\overline{G} \cong \overline{G'}$, we can test the smaller one.

Test 3

Test 3: Isomorphic graphs have the same number of "pieces". We won't define pieces.



Test 4

Definition

The *degree* of a vertex is the number of incident edges.

Note that the sum of the degrees of the vertices is equal to twice $|E|$.

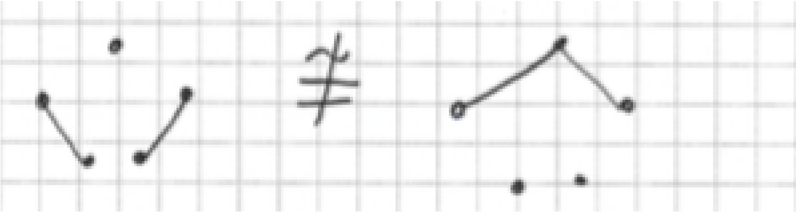
Test 4

Test 4: Isomorphic graphs have the same vertex degree distribution (histogram).

I.e., if $G \cong G'$ and G as m vertices with degree d, then so does G'.

Test 4

Test 4: Isomorphic graphs have the same vertex degree distribution (histogram).



1 vertex of degree 0
4 vertices of degree 1

2 vertices of degree 0
2 vertices of degree 1
1 vertex of degree 2

Test 4

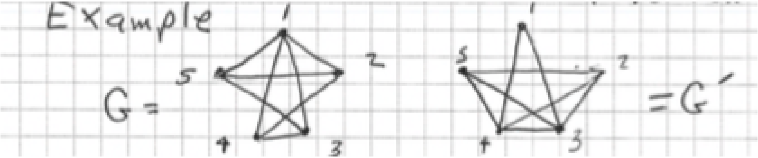
Test 4: Isomorphic graphs have the same vertex degree distribution (histogram).

Exercise for the student: Find two distinguishable graphs which share the same degree distribution.

Test 4

Test 4: Isomorphic graphs have the same vertex degree distribution (histogram).

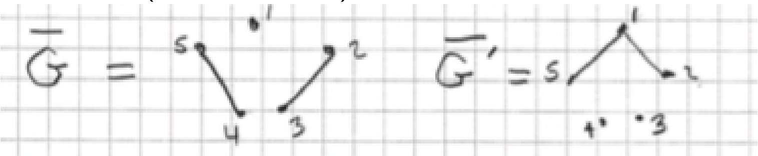
Another example. is $G \cong G'$?



Test 4

Test 4: Isomorphic graphs have the same vertex degree distribution (histogram).

Use test 2 (find $\overline{G}$ and $\overline{G'}$), then either test 3 or test 4.



Test 5

Definition

If $G = (V, E)$ and $G' = (V', E')$ are graphs such that $V' \subset V$ and $E' \subset E$, then G' is called a *subgraph* of G, denoted $G' \subset G$.

Test 5

Recall that a graph is not a set, so this is really new (abuse of) notation $G' \subset G$.

Test 5

Warning, if $G'' = (V'', E'')$ with $V'' \subset V$ and $E'' \subset E$, then G'' is not necessarily a subgraph of G. Example

$$\begin{aligned} G &= \big(\{1,2,3\}, \{\{1,2\}, \{2,3\}\}\big) \\ G'' &= \big(\{1,2\}, \{\{2,3\}\}\big) \end{aligned}$$

Question: G'' is not a subgraph of G. Why?

Test 5

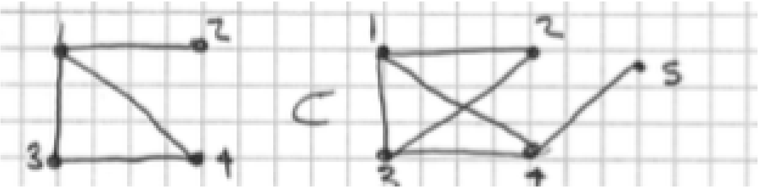
Warning, if $G'' = (V'', E'')$ with $V'' \subset V$ and $E'' \subset E$, then G'' is not necessarily a subgraph of G. Example

$$\begin{aligned} G &= \big(\{1,2,3\},\{\{1,2\},\{2,3\}\}\big) \\ G'' &= \big(\{1,2\},\{\{2,3\}\}\big) \end{aligned}$$

Question: G'' is not a subgraph of G. Why?
Answer: Because G'' is not a graph.

Test 5

Example:



Another example: N_n is a subgraph of every n-vertex graph.

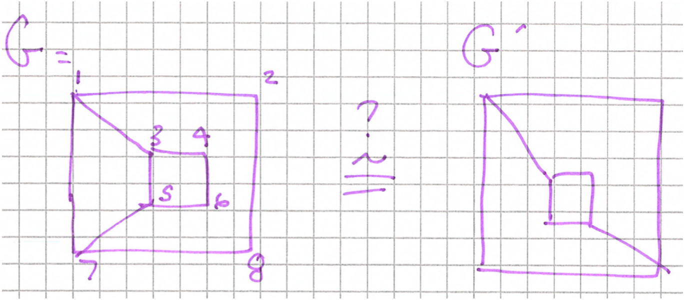
Another example: Every n-vertex graph is a subgraph of K_n.
So $N_n \subset C_n \subset K_n$.

Test 5

Test 5: If $G \cong G'$ and $S \subset G$, then there exists $S' \subset G'$ such that $S \cong S'$.

In other words isomorphic graphs have isomorphic subgraphs.

Test 5

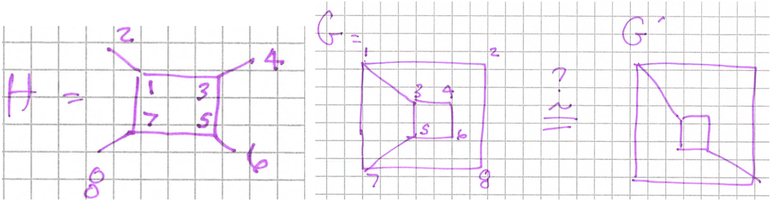


G	G'
10 edges	10 edges
8 vertices	8 vertices
4 vertices of degree 2	4 vertices of degree 2
4 vertices of degree 3	4 vertices of degree 3

We would like to argue that $G \ncong G'$. If we look at tests 1 through 4, we see that none of them helps.

Argument

Let H be the following subgraph of G.



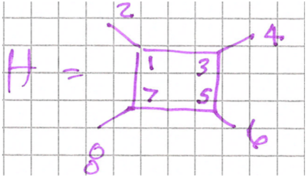
H has the degree distribution:

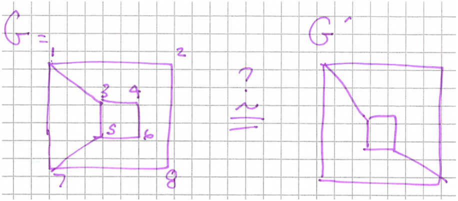
8	vertices
8	edges
4	vertices with degree=3.
4	vertices with degree=1.

We will argue that G' has no such subgraph.

Argument

Let H be the following subgraph of G.





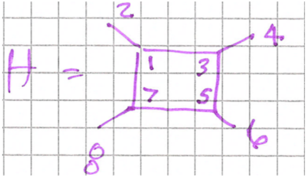
H has the degree distribution:

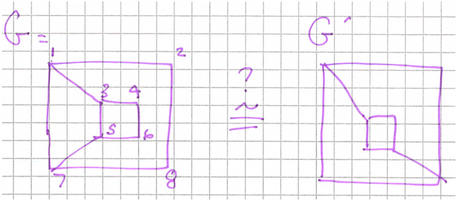
8	vertices
8	edges
4	vertices with degree=3.
4	vertices with degree=1.

G' has 8 vertices and 10 edges. So we must remove 2 edges, and no vertices.

Argument

Let H be the following subgraph of G.





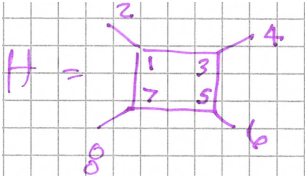
H has the degree distribution:

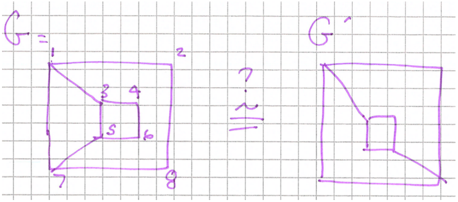
8	vertices
8	edges
4	vertices with degree=3.
4	vertices with degree=1.

G' has 8 vertices and 10 edges. So we must remove 2 edges, and no vertices.
However, G' has 4 vertices of degree 3. So we must remove edges not incident to a 3-degree edge.

Argument

Let H be the following subgraph of G.





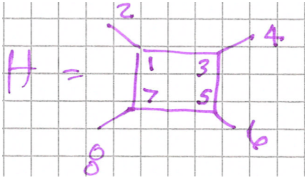
H has the degree distribution:

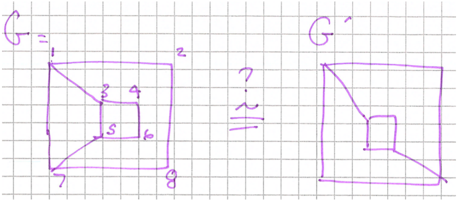
8	vertices
8	edges
4	vertices with degree=3.
4	vertices with degree=1.

However, G' has 4 vertices of degree 3. So we must remove edges not incident to a 3-degree edge.
This is impossible because all edges of G' are adjacent to a 3-degree vertex.

Argument

Let H be the following subgraph of G.





H has the degree distribution:

8	vertices
8	edges
4	vertices with degree=3.
4	vertices with degree=1.

This is impossible because all edges of G' are adjacent to a 3-degree vertex.
Therefore, $H \not\subset G'$, and thus $G \ncong G'$.

End of video