Lecture Notes on Rubik's cube 2x2x2

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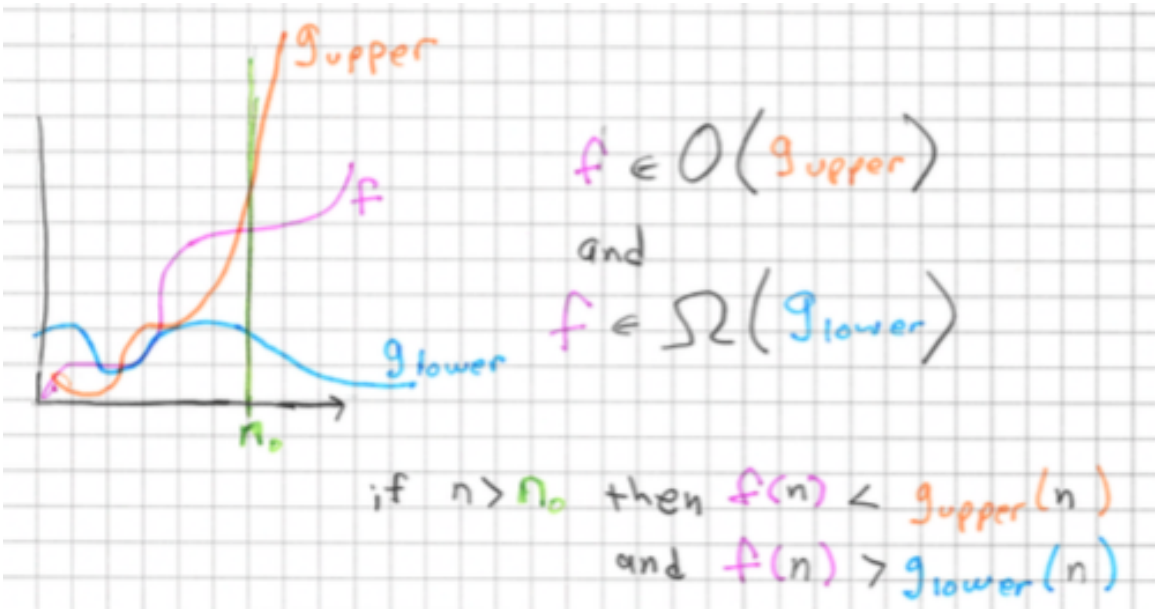
September 22, 2026

1 Formulas to remember

$$\sum_{v \in V} degree(v) = \Theta(|E|)$$

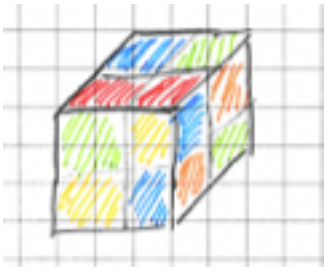
$$|E| = O|V|^2$$

$$|V| = O|E|$$



2 The 2x2x2 Rubiks cube

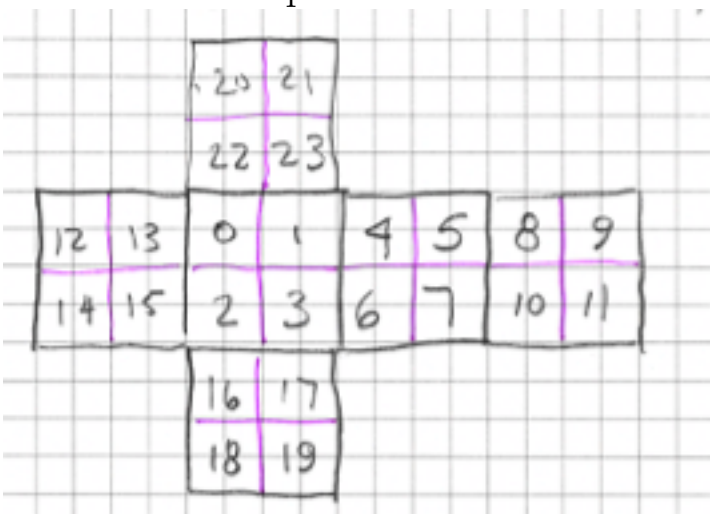
Let's take a look the $2 \times 2 \times 2$ Rubiks cube.



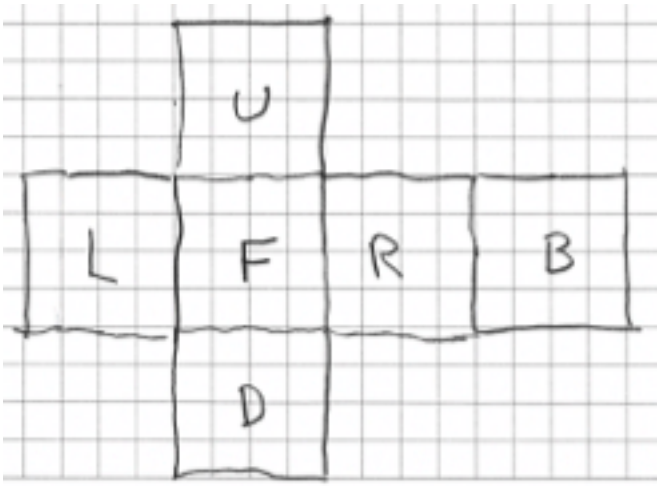
The cube has 6 faces, each having 4 stickers. There are 4 stickers of each color. So $24 = 6 \times 6$ stickers, or positions.

Abbreviation	Color
R	red
W	white
B	blue
G	green
O	orange
Y	yellow

We can represent a state of the cube as an array of size 24. So each of the 24 positions has an index, and the value at that index is the color of the sticker at that position.



To help with notation we denote each of the 6 faces by a single letter name.



Abbreviation	Face
F	front
B	back
U	up
D	down
L	left
R	right

We can represent any state by an appropriate permutation of the string

`WWWWGGGGYYYYBBBBOOOORRRR`

3 Size of state space

How many different states can the cube be in; i.e., what is the size of the state space?

One way to bound the size is by asking what the largest 24 digit base 6 integer is. I.e., if we start with any string representing a state, for example `GWRGYYRBYWBGOROWORB`, and map it to a number according to the following map,

$$
\begin{aligned}
W &\to 0 \\
G &\to 1 \\
Y &\to 2 \\
B &\to 3 \\
0 &\to 4 \\
R &\to 5
\end{aligned}
$$

then we arrive at a 24 digit base 6 number such as

$$\text{BGWRGYYRBYWBGOROWORB} \;\to\; 310512253203145404536 \,.$$

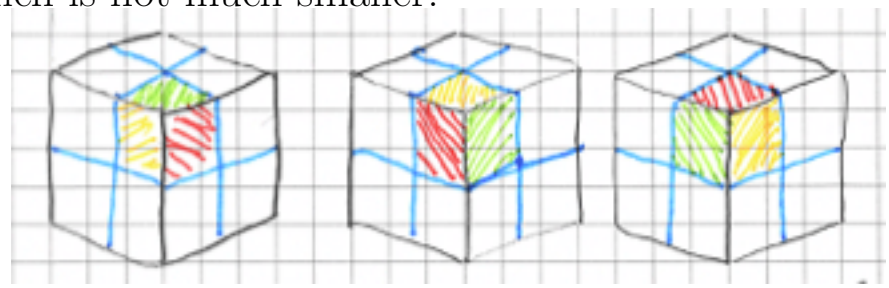
The largest 24 digit base six integer is

$$6^{24} - 1 < 4.73838 \times 10^{18} \,.$$

However, the largest 24 digit base 6 integer we can write with exactly four 5's, four 4's, four 3's, four 2's, four 1's, and four 0's is

$$555544443333222211110000_6 < 4.73765 \times 10^{18} \,,$$

which is not much smaller.



This bound is actually excessively high. How many of these 4.73765×10^{18} potential states are actually reachable by rotations (twists) of the $2 \times 2 \times 2$ Rubik's cube? The cube consists of 8 *cubelets*, each with 3 or stickers or *facelets*. One might imagine any of the cubelets in one of 8 positions, and with any of 3 orientations, $|S| < 8! \times 3^8 = 265,539,520$. We are careful

not to claim that all of these states are reachable by rotating the cube–we are merely trying to bound the size of the state space.

Finally, if we realize that we can rotate the entire cube in our hand to look at any face, without actually twisting the cube, we see that we can look at any of 6 faces, and while looking at the face we can turn the entire cube so that any of the adjacent faces is on top. That means there are $6 \times 4 = 24$ orientations of the cube which we can consider isomorphic. So we can further divide $8! \times 3^8$ by 24 to arrive at

$$\frac{8! \times 3^8}{24} = 11,022,480 < 4.73765 \times 10^{18}\,,$$

which is much less than the upper bounds we derived above.

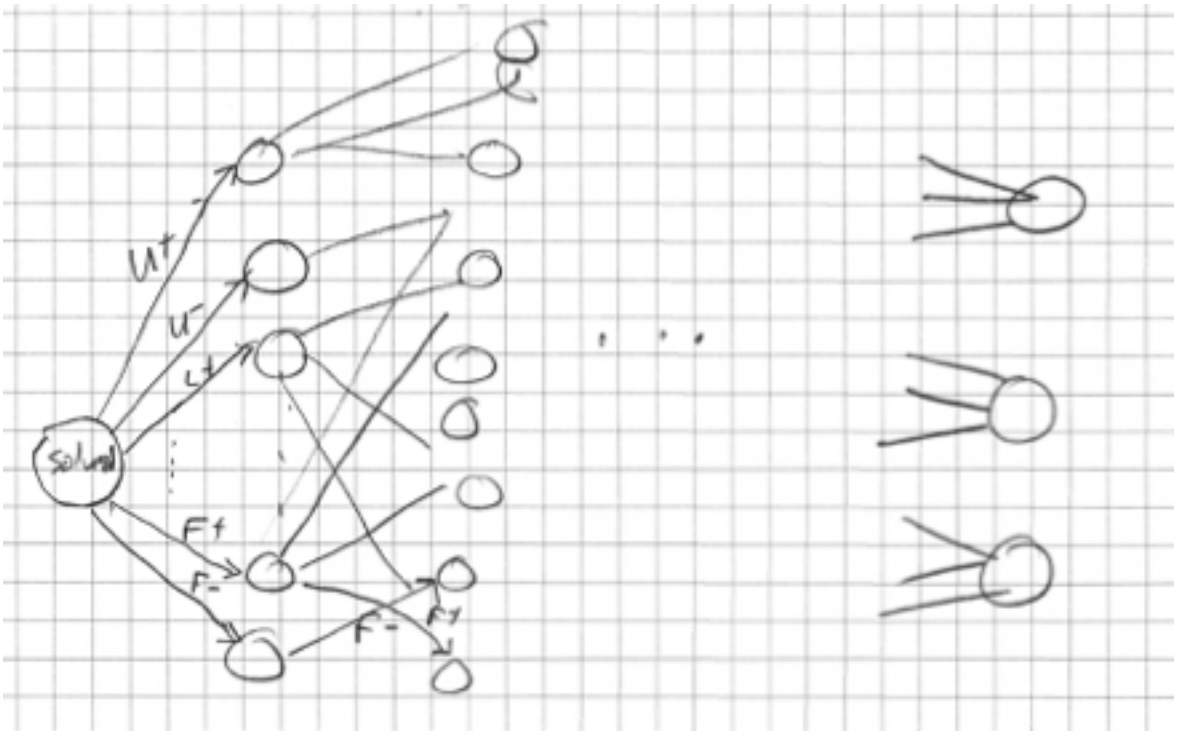
The take-away is that most of what we first considered to be potential states obtainable from permuting the stickers are not actually reachable by twisting the faces of the cube. We will show experimentally, not theoretically, that the actual size of the state space is exactly $\frac{1}{3}$ of this number; $\frac{1}{3} \times \frac{8! \times 3^8}{24} = 3,674,160$.

The $\frac{1}{3}$ is related to the fact that it is impossible to twist one cubelet to any one of its 3 potential orientations without twisting or displacing another–a fact which we will not prove.

The possible rotations (twists) of the cube can be characterized as follows. Each face can be turned either of two directions, 90°

F+	F-	front
B+	B-	back
L+	L-	left
R+	R-	right
U+	U-	up
D+	D-	down

We will consider 180° twists (F2, B2, L2, R2, U2, and D2) later.



Let's define a concept called *frontier*.

Definition 3.1 (frontier 0)**.** The set containing the singleton state of the solved cube, we call this ground state *frontier 0*, denoted Φ_0.

Definition 3.2 (frontier 1)**.** Start from the solved state. The set of states reachable from the solved state by exactly one twist, we call *frontier 1*, and denote it Φ_1.

Definition 3.3 (frontier 2)**.** The set of all states which are reachable from some state in Φ_1 by exactly one twist, excluding any state already encountered, we call this set of states *frontier 2*, and denote it Φ_2.

For example, if we arrive at *frontier 1* by turning the cube F+ we can return to *frontier 0* with the twist F-, because every twist is easily and uniquely reversible.

Definition 3.4 (frontier n)**.** If $n > 0$ we define Φ_n (*frontier n*) as the set of states which are reachable from Φ_{n-1} with a single twist, but are not in $\Phi_0 \cup \Phi_1 \cup ... \cup \Phi_{n-1}$.

We can in this manner bound the sizes of each frontier. $|\Phi_0| = 12^0 = 1$, exactly 1 state, the solved state. Frontier 1 has exactly $|\Phi_1| = 12^1 = 12$, one for each of the 90° twists. Frontier 2, $|\Phi_2| < 12^2 = 144$. Less, because some of the states were seen in $\Phi_0 \cup \Phi_1$, and because some of the 144 are duplicates. For example, we start in Φ_0 and arrive in Φ_2 by F+ F+, or we can arrive at the exact same state by F- F-.

4 God's number

Given two vertices A and B of the graph, a shortest path (maybe not unique), from A to B represents a best sequence of twists to transform the cube from state A to state B.

In particular a shortest path from any vertex v to v_{solved} represents a minimum sequence of twists to solve the cube from that given shuffled state.

What is the maximum such shortest path?

Without 180° twists the answer is 14, and if 180° are allowed, the number is 11. This number is called *God's Number*. It is the number of twists God would need to solve the cube, assuming he always knows the best tactic. God's number for the $3 \times 3 \times 3$ cube was proven in 2010 to be 20. God's number is unknown for any size larger than $3 \times 3 \times 3$.

5 Bounds on a graph

Definition 5.1 (path, path length)**.** Given a graph, $G = (V, E)$, let P be a sequence of *distinct* edges $(e_1, e_2, \ldots, e_n)$ ($e_i \in E$ and no two e_i are identical), such that for each $1 < i < n - 1$ then $e_i \cap e_{i+1} \neq \emptyset$. Note that $e_1 \setminus e_2$ is a set of exactly one vertex; call it u. And $e_n \setminus e_{n-1}$ is a set of exactly one vertex; call it v. Then P is called a *path* from u to v. The number of elements of P is called the *path length*.

Definition 5.2 (distance)**.** Given a graph, $G = (V, E)$, and two vertices, u, and v. There are finitely many paths from u to v. The length of a shortest such path is called the *distance* from u to v, and denoted $\delta(u, v)$.

Definition 5.3 (eccentricity)**.** Given a graph, $G = (V, E)$. If $v \in V$, the *eccentricity* of v is the maximum distance from v. Ie.,

$$\mathit{eccentricity}(v) = \max_{u \in V} \delta(v, u)\,.$$

Definition 5.4 (diameter)**.** Given a graph, $G = (V, E)$. The *diameter* of G is defined by

$$\mathit{diameter}(G) = \max_{v \in V} \mathit{eccentricity}(v)\,.$$

Definition 5.5 (radius)**.** Given a graph, $G = (V, E)$. The *radius* of G is defined by

$$\mathit{radius}(G) = \min_{v \in V} \mathit{eccentricity}(v)\,.$$

Note that by the triangle inequality, the diameter of graph G is bounded by twice the eccentricity of any given vertex.

Claim 1.

$$\mathit{diameter}(G) \leq 2 \times \mathit{radius}(G)$$

Proof. Let D be $\mathit{diameter}(G)$, and $R = \mathit{radius}(G)$.

Let $v_1, v_2 \in V$ such that $\delta(v_1, v_2) = D$.

And take $u \in V$ such that $\mathit{eccentricity}(u) = R$.

By the triangle inequality we have:

$$D \leq \delta(v_1, u) + \delta(u, v_2)$$

We know $\delta(v_1, u) \leq R$ and $\delta(u, v_2) \leq R$, because R is the eccentricity of u, every vertex is within a distance R of u.

Thus

$$D \leq \delta(v_1, v_3) + \delta(v_3, v_2) \leq R + R = 2R\,.$$

□

God's number is the eccentricity of the solved state.

Definition 5.6 (periphery)**.** Given a graph, $G = (V, E)$, the set $P_G \subset V$ of vertices whose eccentricity equal the diameter is called the *periphery*.

$$P_G = \{v \in V \mid \mathit{eccentricity}(v) = \mathit{diameter}(G)\}$$

A vertex in the periphery of a graph is called a *peripheral vertex*.

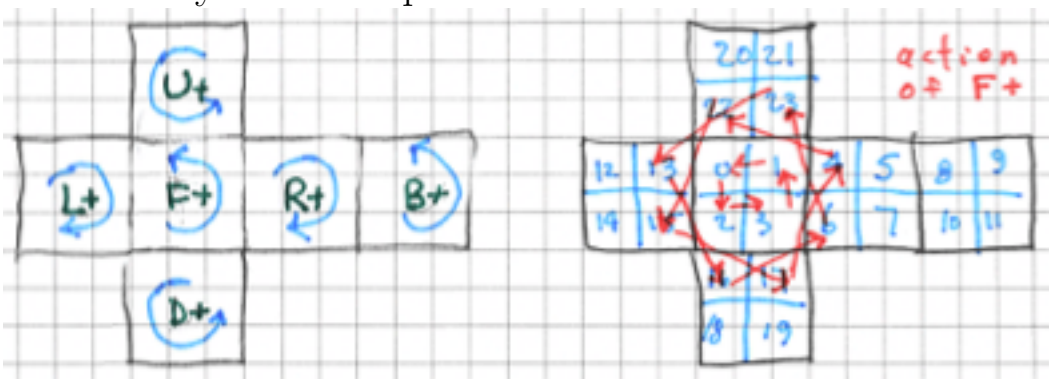
Definition 5.7 (center, central vertex)**.** Given a graph, $G = (V, E)$, the set $C_G \subset V$ of vertices whose eccentricity equal the radius is called the *center*.

$$C_G = \{v \in V \mid \mathit{eccentricity}(v) = \mathit{radius}(G)\}$$

A vertex in the center of a graph is called a *central vertex*.

6 Implementing the Rubik's cube

Recall cycle notation for specifying a permutation. A permutation that transforms (123456) to (421365) can be denoted as (143)(56), meaning $1 \mapsto 4 \mapsto 3 \mapsto 1$ and $5 \mapsto 6 \mapsto 5$. We can represent a twist of the cube in terms of the cycles of its permutation of the facelets.



We can represent each of the forward 90° twists as the following cycles.

F+	=	(0 2 3 1)	(4 22 15 17)	(6 23 13 16)
B+	=	(5 19 14 20)	(7 18 12 21)	(8 10 11 9)
L+	=	(0 16 11 20)	(2 18 9 22)	(12 13 15 14)
R+	=	(1 21 10 17)	(3 23 8 19)	(4 5 7 6)
U+	=	(0 4 8 12)	(1 5 9 13)	(20 22 23 21)
D+	=	(2 14 10 6)	(3 15 11 7)	(16 18 19 17)

The inverse rotations, can be formed by reversing the lists.

F-	=	(1 3 2 0)	(17 15 22 4)	(16 13 23 6)
B-	=	(20 14 19 5)	(21 12 18 7)	(9 11 10 8)
L-	=	(20 11 16 0)	(22 9 18 2)	(14 15 13 12)
R-	=	(17 10 21 1)	(19 8 23 3)	(6 7 5 4)
U-	=	(12 8 4 0)	(13 9 5 1)	(21 23 22 20)
D-	=	(6 10 14 2)	(7 11 15 3)	(17 19 18 16)

The 180° rotations are just double applications of the 90° twists.

F2	=	(1 3 2 0)	(17 15 22 4)	(16 13 23 6)
		(1 3 2 0)	(17 15 22 4)	(16 13 23 6)
B2	=	(20 14 19 5)	(21 12 18 7)	(9 11 10 8)
		(20 14 19 5)	(21 12 18 7)	(9 11 10 8)
L2	=	(20 11 16 0)	(22 9 18 2)	(14 15 13 12)
		(20 11 16 0)	(22 9 18 2)	(14 15 13 12)
R2	=	(17 10 21 1)	(19 8 23 3)	(6 7 5 4)
		(17 10 21 1)	(19 8 23 3)	(6 7 5 4)
U2	=	(12 8 4 0)	(13 9 5 1)	(21 23 22 20)
		(12 8 4 0)	(13 9 5 1)	(21 23 22 20)
D2	=	(6 10 14 2)	(7 11 15 3)	(17 19 18 16)
		(6 10 14 2)	(7 11 15 3)	(17 19 18 16)

There is an important optimization which may not be obivous. It turns out we can obtain every state of the cube by combinations which only make use of 6 twists. Why? For example, we can effectuate the L+ twist by performing R- and then rolling the entire cube toward myself. You can think of this as: R+L- simply rotates the entire cube. Similar

for the other two pairs of opposite sides. Therefore we can obtain any permutation of the cube simply using D-, D+, R-, R+, B-, and B+. Or if we want to encorporate the 180° degree twists, we can extend this set of 6 twists to 9 twists D-, D+, D2, R-, R+, R2, B-, B+, and B2. This optimization greatly reduces the amount of memory needed to run such a simulation as a computer program.

7 Breadth first search BFS

We can examine the Rubik's cube graph using the BFS algorithm.

Algorithm 1: BFS(s,Adj)

Input : s a starting vertex
Input : Adj graph adjacency list, as a mapping

```
1  level[s] ← 0
2  Π[s] ← None
3  fifo.push(s)
4  while fifo not empty do
5  |  v ← fifo.pop()
6  |  for u ∈ Adj[v] do
7  |  |  if not Π[u] then
8  |  |  |  Π[u] ← v
9  |  |  |  level[u] ← 1 + level[v]
10 |  |  |  fifo.push(u)
```

To analyze the complexity, notice that every reachable edge goes into the fifo exactly once (push), and comes out (pop) exactly once. The `while` loop repeats $|V|$ times. The complexity is

$$O(|V| + |E|)$$

The key insight is that *for* $u \in Adj[v]$ occurs once per edge (or twice for an undirected graph). Recall

$$\sum_{v \in V} degree(v) = 2|E| = \Theta(|E|)$$

As an alternate to the fifo approach to BFS, we can also use two stacks, one for pushing and one for popping.

Algorithm 2: BFS(s,Adj)

Input : s a starting vertex

Input : Adj graph adjacency list, as a mapping

```
1  level[s] ← 0
2  eccentricity ← 0
3  Π[s] ← None
4  frontier ← empty stack
5  frontier.push(s)
6  while frontier not empty do
7  |  next ← empty stack
8  |  for v ∈ frontier do
9  |  |  for u ∈ Adj[v] do
10 |  |  |  if not Π[u] then
11 |  |  |  |  next.push(u)
12 |  |  |  |  Π[u] ← v
13 |  |  |  |  level[u] ← eccentricity
14 |  |  frontier ← next
15 |  |  eccentricity = 1 + eccentricity
```